Prime gaps, probabilistic models, the interval sieve, Hardy-Littlewood conjectures and Siegel zeros

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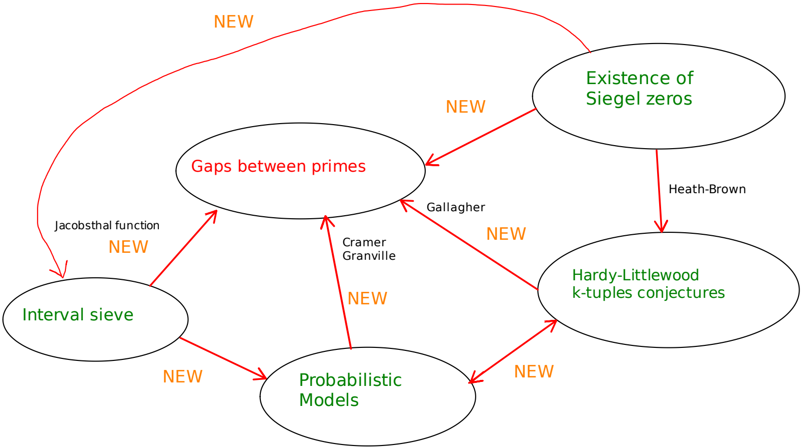
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Existence of
Siegel zeros
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Gaps between primes
Heath-Brown
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NEW

$G(x) := \max_{p_n \leqslant x} (p_n - p_{n-1})$, p_n is the n^{th} prime.

Upper bound: $G(x) = O(x^{0.525})$ (Baker-Harman-Pintz, 2001).
Improve to $O(x^{1/2} \log x)$ on Riemann Hypothesis (Cramér, 1920).

Lower bound: $G(x) \gg (\log x) \dfrac{\log\log x \log\log\log\log x}{\log\log\log x}$

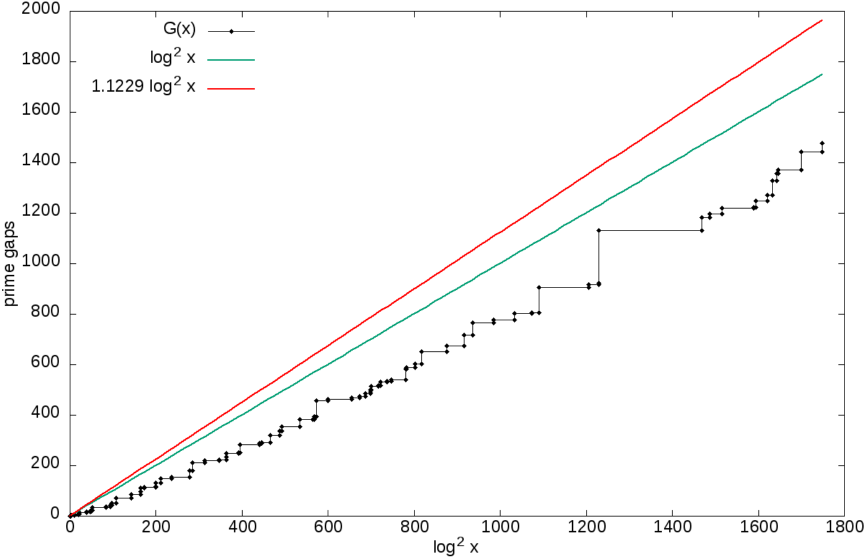
(F,Green,Konyagin,Maynard,Tao,2018)

Cramér (1936) conjecture: $\limsup_{x\to\infty} \dfrac{G(x)}{\log^2 x} = 1.$

Granville (1995) conjecture: $\limsup_{x\to\infty} \dfrac{G(x)}{\log^2 x} \geqslant 2e^{-\gamma} = 1.1229\ldots$

Computations: $\sup_{x \leqslant 10^{18}} \dfrac{G(x)}{\log^2 x} \approx 0.92.$

Computational evidence, up to 10^{18}



Siegel zeros and large gaps

Idea #1. Jacobsthal's function (Westzynthius-Erdős-Rankin). Define

$$\mathcal{U}_T = \{n \in \mathbb{Z} : n \text{ has no prime factor } \leqslant T\}.$$

If $\mathcal{U}_T$ has a gap of size y then $G(2Q) \geqslant y$, where $Q = \prod_{p \leqslant T} p \approx e^T$.

Idea # 2. Gallagher's PNT, 1970. If $q \in \mathbb{N}$, χ is a nonprincipal, real Dirichlet character mod q, and

$$L(1-\delta, \chi) = 0 \qquad \text{(A)}$$

then for $x \geqslant q^B$ (B is an absolute const.),

$$\pi(x; q, 1) := \#\{p \leqslant x : p \equiv 1 \pmod{q}\} \ll \frac{\delta x}{\phi(q)}.$$

Siegel. $\delta \gg_\varepsilon q^{-\varepsilon}$, ineffective constants!

Theorem. (F 2019)

Assume (A). Then $G(X) \gg \left(\frac{1}{\delta}\right) \frac{\log X}{\log\log X}$, $\quad X = \exp(q^C)$, C constant.

Siegel zeros and large gaps, II

Theorem. (F 2019)

Assume (A). Then $G(X) \gg \left(\frac{1}{\delta}\right) \frac{\log X}{\log\log X}$, $\quad X = \exp(q^C)$, C constant.

Examples

(1) If $\delta \leqslant (\log q)^{-K}$ for infinitely many q, then infinitely often we have $G(X) \gg (\log X)(\log\log X)^{K-1}$.

(2) If $\delta \leqslant q^{-\varepsilon(q)}, \varepsilon(q) \downarrow 0$ slowly, for infinitely many q, then infinitely often, $G(X) \gg (\log X)^{1+\delta(X)}, \delta(X) \downarrow 0$ slowly.

Unconditionally:

$$G(X) \gg (\log X) \frac{\log\log X \log\log\log\log X}{\log\log\log X} = (\log X)(\log\log X)^{1+o(1)}.$$

Siegel zeros and large gaps, III

Theorem. (F 2019)

Assume (A). Then $G(X) \gg \left(\frac{1}{\delta}\right) \frac{\log X}{\log\log X}, \quad X = \exp(q^C)$, C constant.

Proof sketch. Assume q is prime, $B \geqslant 4$. Take $T = (C/2)\delta q^{B-1} \log q$, $x = 2q^B$, $y = \frac{x}{2q} = q^{B-1}$. For $p \leqslant T/2$, $p \nmid q$ take a_p s.t. $qa_p + 1 \equiv 0 \pmod{p}$. If $0 \leqslant n \leqslant y$ is not covered by $\{a_p \mod p : p \leqslant T/2\}$, then $qn+1$ has no prime factor $\leqslant T/2$. But $qn + 1 \ll q^B \ll T^{5/3}$ (by Siegel's theorem), so $qn + 1$ is prime. By Gallagher, there are at most

$$\pi(x; q, 1) \leqslant C_1 \frac{\delta x}{q} \leqslant \frac{T}{10 \log T}$$

elements of $[0, y]$ left uncovered (if C is large). Cover these using $p \in (T/2, T]$. Thus, $\{a_p \mod p : p \leqslant T\}$ covers $[0, y]$, giving $G(e^{2T}) \geqslant y$ as required.

Cramér's model of large prime gaps.

Random set $\mathcal{C} = \{C_1, C_2, \ldots\} \subset \mathbb{N}$, choose $n \geqslant 3$ to be in $\mathcal{C}$ with prob. $1/\log n$. matching the density of primes near n by the PNT.

Theorem. (Cramér 1936)

With probability 1,

$$\limsup_{m\to\infty} \frac{C_{m+1} - C_m}{\log^2 C_m} = 1.$$

Cramér: "*for the ordinary sequence of prime numbers p_n, some similar relation may hold*".

Proof:

$$\mathbb{P}(n+1, \ldots, n+k \notin \mathcal{C}) \sim \left(1 - \frac{1}{\log n}\right)^k \sim e^{-k/\log n}.$$

if $k > (1+\varepsilon)\log^2 n$, this is $\ll n^{-1-\varepsilon}$. Sum over n converges.
if $k < (1-\varepsilon)\log^2 n$, this is $\gg n^{-1+\varepsilon}$. Sum over n diverges.
Finish with the Borel-Cantelli lemma.

Cramér model vs. primes: the good, the bad and the ugly

The good: Cramér 1936. "Probabilistic RH"

With probability 1, $\#\{n \leqslant x : n \in \mathcal{C}\} = \text{li}(x) + O(x^{1/2+\varepsilon})$.

the bad: twin primes in the model

With probability 1, $\#\{n \leqslant x : n, n+2 \in \mathcal{C}\} \sim \frac{x}{(\log x)^2}$.

Hardy-Littlewood predicts $\#\{n \leqslant x : n, n+2 \text{ prime}\} \sim 1.32\ldots \frac{x}{\log^2 x}$.

the ugly: gaps of size 1

With probability 1, $\#\{n \leqslant x : n, n+1 \in \mathcal{C}\} \sim \frac{x}{(\log x)^2}$.

Cramér: For finite $\mathcal{H}$, $\#\{n \leqslant x : n+h \in \mathcal{C}\ \forall h \in \mathcal{H}\} \sim \frac{x}{(\log x)^{|\mathcal{H}|}}$.

Prime k-tuples Conjecture (Hardy-Littlewood, 1922)

$$\#\{n \leqslant x : n+h \text{ prime } \forall h \in \mathcal{H}\} = \big(\mathfrak{S}(\mathcal{H}) + o(1)\big)\frac{x}{(\log x)^{|\mathcal{H}|}} \quad (x \to \infty),$$

where $\mathfrak{S}(\mathcal{H})$ is the "singular series"

$$\mathfrak{S}(\mathcal{H}) := \prod_p \left(1 - \frac{|\mathcal{H} \bmod p|}{p}\right)\left(1 - \frac{1}{p}\right)^{-|\mathcal{H}|}$$

The factor $\mathfrak{S}(\mathcal{H})$ captures the bias of real primes; $\mathcal{H}$ avoids $0 \mod p$ for all p.

- $\mathcal{H} = \{0\}$. $\mathfrak{S}(\mathcal{H}) = 1$. Prime Number Theorem;
- $\mathcal{H} = \{0, 2\}$. $\mathfrak{S}(\mathcal{H}) = 1.32\ldots$, twin prime constant.
- $\mathcal{H} = \{0, 1\}$. $\mathfrak{S}(\mathcal{H}) = 0$.

$\mathcal{U}_T := \{n \in \mathbb{Z} : \text{no prime factor } \leqslant T\}$, density$(\mathcal{U}_T) = \theta := \prod_{p \leqslant T} (1 - 1/p)$

Granville's random set $\mathcal{G} = \{G_1, G_2, \ldots\}$:

Set $T = (\log x)^{1-o(1)}$. Take $n \in (x, 2x]$ with probability $\begin{cases} 0 & \text{if } n \notin \mathcal{U}_T \\ \frac{1/\theta}{\log n} & \text{if } n \in \mathcal{U}_T. \end{cases}$.

***k*-tuples.** With prob. 1, $\#\{n \leqslant x : n + h \in \mathcal{G}\ \forall h \in \mathcal{H}\} \sim \mathfrak{S}(\mathcal{H}) \frac{x}{(\log x)^{|\mathcal{H}|}}$.

Theorem. (Granville 1995)

With prob. 1, $\limsup_{n\to\infty} \frac{G_{n+1} - G_n}{\log^2 G_n} \geqslant 2e^{-\gamma} = 1.1229\ldots$

Idea: with $y = c \log^2 x = T^{2+o(1)}$, $\#\left([0, y] \cap \mathcal{U}_T\right) \sim \frac{y}{\log y}$.

By contrast, for a *typical* $a \in \mathbb{Z}$,

$$\#\left([a, a+y] \cap \mathcal{U}_T\right) \sim \theta y \sim 2e^{-\gamma} \frac{y}{\log y} \qquad \text{(Mertens)}$$

Hardy-Littlewood conjecture (standard version)

$$\#\{n \leqslant x : n+h \text{ prime } \forall h \in \mathcal{H}\} \sim \mathfrak{S}(\mathcal{H}) \frac{x}{(\log x)^{|\mathcal{H}|}}.$$

$$\mathfrak{S}(\mathcal{H}) \approx \mathfrak{S}_z(\mathcal{H}) := \underbrace{\prod_{p \leqslant z} \left(1 - \frac{|\mathcal{H} \bmod p|}{p}\right)}_{=\mathbb{P}(\mathcal{H} \subset \mathcal{S}_z)} \underbrace{\prod_{p \leqslant z} \left(1 - \frac{1}{p}\right)^{-|\mathcal{H}|}}_{\approx (e^\gamma \log z)^{|\mathcal{H}|}}.$$

Take $a_p \in \{0, \ldots, p-1\}$ at random for each prime p. With the a_p fixed, define

$$\mathcal{S}_z = \{n \in \mathbb{Z} : n \not\equiv a_p \pmod p, p \leqslant z\}. \quad \text{“random sieve”.}$$

Pólya’s magic exponent: $z = z(x) \sim x^{1/e^\gamma}$.

Hardy-Littlewood conjecture (probabilistic version)

$$\frac{1}{x} \#\{n \leqslant x : n+h \text{ prime } \forall h \in \mathcal{H}\} \sim \mathbb{P}(\mathcal{H} \subset \mathcal{S}_{z(x)}).$$

Hardy and Littlewood (1923, Partitio numerorum III)

"Probability is not a notion of pure mathematics, but of philosophy or physics."

George Polya (1959)

"I do not pretend to understand why the introduction of the upper bound x^{1/e^γ} should yield the right result.

For that matter, when the quanta were introduced, no physicist pretended to understand why energy should be obtainable (as salt or sugar is in the self-service store) only in uniform little packages, in multipla of a certain unit. Yet the criterion of a physical theory is its applicability. "

A new "random sieve" model of primes (Banks,F,Tao)

More precise def: $z(x)$ is the largest z with $\prod_{p \leqslant z}(1 - 1/p) \geqslant 1/\log x$.

Mertens + PNT: $z(x) \sim x^{1/e^\gamma}$.

Random set $\mathcal{R} = \{n \geqslant 3 : n \notin \mathcal{S}_{z(n)}\}$.

Global density: $\mathbb{P}(n \in \mathcal{R}) = \mathbb{P}(n \notin \mathcal{S}_{z(n)}) = \prod_{p \leqslant z(n)}(1 - 1/p) \sim \frac{1}{\log n}$. Matches primes.

Difficulty: $n_1 \in \mathcal{R}$, $n_2 \in \mathcal{R}$ are very dependent.

We conjecture that the primes and $\mathcal{R}$ share similar *local statistics*.

Hardy-Littlewood statistics for arbitrary sequences

Let $\mathcal{A} \subset \mathbb{N}$. For functions $y = y(x), K = K(x)$, we say that $\mathcal{A}$ satisfies Hypothesis $\mathrm{HL}(\mathcal{A}; y, K, c)$ if uniformly for all $\mathcal{H} \subset [0, y]$ with $|\mathcal{H}| \leqslant K$ we have

$$\#\{n \leqslant x : n + h \in \mathcal{A} \ \forall h \in \mathcal{H}\} = \int_2^x \mathbb{P}(\mathcal{H} \subset \mathcal{S}_{z(t)})\, dt + O(x^c).$$

Strong Hardy-Littlewood conjecture

For all $\varepsilon > 0$, Hypothesis $\mathrm{HL}(\text{Primes}; \sqrt{x}, \log x, 1/2 + \varepsilon)$ holds.

Theorem. (BFT 2019+)

Fix $\frac{1}{2} \leqslant c < 1$, $\varepsilon > 0$. With probability 1, we have

$$\mathrm{HL}(\mathcal{R}; \exp\{(\log x)^{c-\varepsilon}\}, (\log x)^c, \delta(c) + \varepsilon),$$

where $\delta(1/2) = \frac{1}{2}$ and $\delta(c) < 1$ for $1/2 < c < 1$.

Gallagher: HL implies Poisson gaps

Theorem. (Gallagher 1976)

Assume $\#\{n \leqslant x : n+h \text{ prime } \forall h \in \mathcal{H}\} \sim \mathfrak{S}(\mathcal{H}) \int_2^x \frac{dt}{\log^{|\mathcal{H}|} t}$ uniformly for $|\mathcal{H}| \leqslant k$ (k fixed) and $\mathcal{H} \subset [0, \log^2 x]$. Then

$$\#\{n \leqslant x : p_{n+1} - p_n > \lambda \log x\} \sim e^{-\lambda} \pi(x) \quad (\lambda > 0 \text{ fixed}).$$

Main tool:

$$\sum_{\substack{\mathcal{H} \subset [0,y] \\ |\mathcal{H}|=k}} \mathfrak{S}(\mathcal{H}) \sim y^k / k!. \tag{1}$$

Montgomery-Soundararajan (2004) gave further terms in (1). Poor uniformity in k.

Gallagher's argument works for any sequence, not just primes.

Hardy-Littlewood implies large gaps

Theorem. (BFT 2019. Uniform Hardy Littlewood implies large gaps)

Assume $\frac{2\log\log x}{\log x} \leqslant \kappa \leqslant 1/2$ and that $\mathrm{HL}(\mathcal{A}; \log^2 x; \frac{\kappa \log x}{2\log\log x}; 1-\kappa)$ holds. Then

$$G_{\mathcal{A}}(x) := \max\{b - a : 1 \leqslant a < b \leqslant x, (a, b] \cap \mathcal{A} = \emptyset\} \gg \frac{\kappa(\log x)^2}{\log\log x}.$$

Corollary

If HL(Primes; $\log^2 x$; $\frac{\log x}{8\log\log x}$; $3/4$) holds, then $G(x) \gg \frac{(\log x)^2}{\log\log x}$.

Corollary

For any $\varepsilon > 0$, with probability 1 we have $G_{\mathcal{R}}(x) \gg (\log x)^{2-\varepsilon}$.

Large gaps from Hardy-Littlewood

Proof sketch: Weighted count of gaps of size $\geqslant y$:

$$\#\{n \leqslant x : [n, n+y] \cap \mathcal{A} = \emptyset\} = \sum_{n \leqslant x} \underbrace{\prod_{0 \leqslant h \leqslant y} (1 - \mathbf{1}_{n+h \in \mathcal{A}})}_{\text{gap detector}}$$

$$= \sum_{k=0}^{y} (-1)^k \sum_{\substack{\mathcal{H} \subset [0,y] \\ |\mathcal{H}|=k}} \underbrace{\sum_{n \leqslant x} \prod_{h \in \mathcal{H}} \mathbf{1}_{\mathcal{A}}(n+h)}_{\text{HL assumption}}$$

$$\approx \sum_{k=0}^{y} (-1)^k \sum_{\substack{\mathcal{H} \subset [0,y] \\ |\mathcal{H}|=k}} \int_2^x \mathbb{P}(\mathcal{H} \subset \mathcal{S}_{z(t)})\, dt$$

$$= \int_2^x \mathbb{E} \sum_{k=0}^{y} (-1)^k \binom{|\mathcal{S}_{z(t)} \cap [0,y]|}{k}\, dt$$

$$= \int_2^x \mathbb{P}(\mathcal{S}_{z(t)} \cap [0,y] = \emptyset)\, dt.$$

Large gaps in $\mathcal{A} \longleftrightarrow$ Large gaps in $\mathcal{S}_z$

HL implies no super-large gaps?

Does a uniform HL for $\mathcal{A}$ imply an *upper bound* on large gaps?

Answer: NO!

Removal of all elements of $\mathcal{A}$ in an interval $(y, y + \sqrt{y})$, for an infinite sequence of y's, does not affect the HL statistics but creates a very large gap.

Large gaps in our new model: interval sieve

Interval sieve extremal bound

Let $z = (y/\log y)^{1/2}$ and define

$$W_y := \min_{(a_p)} \big|[0,y] \cap \mathcal{S}_z\big| = \min_u \#\Big\{n \in (u, u+y] : n \text{ has no prime factor } \leqslant z\Big\}.$$

Known bounds: $\underbrace{\frac{4y \log\log y}{\log^2 y}}_{\text{Iwaniec, lin. sieve}} \lesssim W_y \lesssim \underbrace{\frac{y}{\log y}}_{u=0}$. **Folklore conj:** $W_y \sim \frac{y}{\log y}$

Theorem (BFT, 2019)

If "Siegel zeros exist" ($\exists$ infinite sequence of zeros β_q of quadratic Dirichlet L-functions modulo q, with $\beta_q = o(1/\log q)$) then for a sequence $y_j \to \infty$,

$$W_{y_j} = o(y_j/\log y_j).$$

New model and large gaps

Def: $G_{\mathcal{R}}(x)$ is largest gap between consec. elements of $\mathcal{R}$ that are $\leqslant x$.

$$W_y := \min \big| [0,y] \cap \mathcal{S}_{(y/\log y)^{1/2}} \big|. \qquad g(u) := \max\{y : W_y \log y \leqslant u\}.$$

Then

$$\frac{4y\log\log y}{\log^2 y} \lesssim W_y \lesssim \frac{y}{\log y} \quad \Rightarrow \quad u \lesssim g(u) \lesssim \frac{u \log u}{4 \log\log u}.$$

Theorem. (BFT 2019)

For all $\varepsilon > 0$, with probability 1 there is x_0 s.t.

$$g((2e^{-\gamma} - \varepsilon)\log^2 x) \leqslant G_{\mathcal{R}}(x) \leqslant g((2e^{-\gamma} + \varepsilon)\xi \log^2 x) \quad (x > x_0).$$

Proof tools: Small sieve, large sieve, large deviation inequalities (Bennett's inequality, Azuma's martingale inequality), combinatorics, ...

We have the same bounds for Granville's model, that is, for $G_{\mathcal{G}}(x)$.

New model and large gaps

Theorem. (BFT 2019)

For all $\varepsilon > 0$, with probability 1 there is x_0 s.t.

$$g((2e^{-\gamma} - \varepsilon)\log^2 x) \leqslant G_{\mathcal{R}}(x) \leqslant g((2e^{-\gamma} + \varepsilon)\log^2 x) \quad (x > x_0).$$

Conjecture. (BFT 2019)

For the largest gap $G(x)$ between primes $\leqslant x$,

$$G(x) \sim g(2e^{-\gamma}\log^2 x) \qquad (x \to \infty).$$

If this is true, then $2e^{-\gamma}\log^2 x \lesssim G(x) \lesssim 2e^{-\gamma}\log^2 x \left(\frac{\log\log x}{2\log\log\log x}\right)$.

If Siegel zeros exist, our model predicts $\limsup_{x\to\infty} \frac{G(x)}{\log^2 x} = \infty$.