THE DISTRIBUTION OF DIVISORS OF POLYNOMIALS

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ABSTRACT. Let $F(x)$ be an irreducible polynomial with integer coefficients and degree at least 2. For $x \geqslant z \geqslant y \geqslant 2$, denote by $H_F(x, y, z)$ the number of integers $n \leqslant x$ such that $F(n)$ has at least one divisor d with $y < d \leqslant z$. We determine the order of magnitude of $H_F(x, y, z)$ uniformly for $y + y/\log^C y < z \leqslant y^2$ and $y \leqslant x^{1-\delta}$, showing that the order is the same as the order of $H(x, y, z)$, the number of positive integers $n \leqslant x$ with a divisor in $(y, z]$. Here C is an arbitrarily large constant and $\delta > 0$ is arbitrarily small.

1. Introduction

Let $F(t) \in \mathbb{Z}[t]$ be an irreducible polynomial of degree $g \geqslant 2$. In this paper we study the size of $H_F(x, y, z)$, the number of positive integers $n \leqslant x$ for which $F(n)$ has a divisor in $(y, z]$. The special case $F(t) = t$, counting integers $n \leqslant x$ with a divisor in $(y, z]$, is classical and goes back to early work of Besicovitch and Erdős in the 1930s. In 2008, the first author [**7**] determined the exact order of growth of $H(x, y, z)$ for all x, y, z. In particular, we have

$$H(x, y, 2y) \asymp \frac{x}{(\log y)^{\mathcal{E}}(\log\log y)^{3/2}} \qquad (10 \leqslant y \leqslant \sqrt{x}), \tag{1.1}$$

where

$$\mathcal{E} = 1 - \frac{1 + \log\log 2}{\log 2} = 0.086071332\ldots.$$

The corresponding estimate for a linear polynomial F follows from an argument identical to that in [**7**], uniformly in the coefficients (see e.g., Proposition 2 in [**10**]). The study of $H_F(x, y, z)$ for a general polynomial began in connection with the problem of bounding from below the largest prime factor of $\prod_{n \leqslant x} F(n)$. This problem began with work of Chebyshev (see Markov [**22**]) for $F(t) = t^2 + 1$ and has received a great deal of attention since. For work on bounding the largest prime factor of $\prod_{n \leqslant x} F(n)$ for specific polynomials F, see the important papers of Ivanov [**15**], Hooley [**12**], Hooley [**13**], Deshouillers and Iwaniec [**2**], Heath-Brown [**11**], Irving [**14**], Dartyge [**1**], la Bretèche [**18**], la Bretèche and Drappeau [**19**] and Merikowski [**23**]. The first bound on the largest prime factor of $\prod_{n \leqslant x} F(n)$ for general F is due to Nagell [**24**], and was subsequently

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improved by Erdős [**4**], Erdős and Schinzel [**5**], and most recently by Tenenbaum [**28**]. Erdős and Schinzel [**5**] gave the explicit lower bound

$$\max\Big\{p : p\Big|\prod_{n\leqslant x} F(n)\Big\} \gg x\exp\Big\{\frac{\log x}{gx} H_F(x, \frac{x}{2}, x)\Big\},$$

where g is the degree of F (this bound is also implicit in Erdős [**4**]). The best lower bounds for $H_F(x, x/2, x)$ are due to Tenenbaum [**28**], who showed that

$$H_F(x, x/2, x) \gg_F x/(\log x)^{\log 4 - 1 + o(1)} \qquad (x \to \infty). \tag{1.2}$$

In [**27**], Tenenbaum took up the problem of bounding $H_F(x, y, z)$ for general x, y, z. There are technical difficulties that arise when $y \gg x$, and thus Tenenbaum restricted his attention to the case $y \leqslant x^{1-\delta}$ for some fixed $\delta > 0$. In this case he proved the following (we compare with the size of $H(x, y, z)$, as the order is now known).

Theorem T1. *Let $\delta > 0$ and $C > 1$ be real. Then if y_0 is large enough, depending only on δ, C, F, and also $y_0 \leqslant y \leqslant x^{1-\delta}$ and $y + y/(\log y)^C \leqslant z \leqslant y^2$, then*

$$H_F(x, y, z) = H(x, y, z)\exp\{O_{\delta,C,F}(\sqrt{\log\log y \log\log\log y})\}.$$

In particular, combined with (1.1) we see that $H_F(x, y, 2y) = x(\log y)^{-\mathcal{E}+o(1)}$ uniformly for $y_0 \leqslant y \leqslant x^{1-\delta}$. Tenenbaum's paper deals with arbitrary polynomials, irreducible or reducible. In order to remove various technical issues that pertain to reducible polynomials, we focus here on the irreducible case. We record here only one of the more important estimates of Tenenbaum in the reducible case; see (1.13) in [**27**].

Theorem T2. *Let $F \in \mathbb{Z}[x]$ be a reducible polynomial which factors as $F(x) = \prod_{j=1}^r F_j(x)^{\alpha_j}$, where $F_1, \ldots, F_r$ are distinct and irreducible. Define $\tau = -1 + \sum_j \log(\alpha_j + 1)$. For any $\delta > 0$, there is a constant C so that uniformly for $y + y/(\log y)^{\tau-\delta} \leqslant z$ and $y \leqslant x^{1-\delta}$ we have $H_F(x, y, z) \asymp x$, the implied constants depending on F, δ.*

Let $\rho(d)$ be the number of solutions of $F(n) \equiv 0 \pmod d$. Heuristically, we expect $H_F(x, y, z)$ to behave like $H(x, y, z)$ since the average of $\rho(p)$ over primes p is 1. Consequently, the distribution of the the prime factors of $F(n)$, over a randomly chosen $n \leqslant x$, should be very close to the distribution of the prime factors of that for a randomly chosen $n \leqslant x$. We confirm this heuristic in the same range of the variables as in Theorem T1. In order to facilitate future applications, we state a lower bound for the number of $n \in (x/2, x]$ with $F(n)$ having a divisor in $(y, z]$.

Theorem 1. *Let $F(t) \in \mathbb{Z}[t]$ be irreducible. Let $\delta > 0$ be an arbitrarily small positive constant, and $C > 1$ an arbitrarily large constant. For some sufficiently large $y_0 = y_0(F, \delta, C)$, we have*

$$H_F(x, y, z) \ll H(x, y, z) \ll H_F(x, y, z) - H_F(x/2, y, z)$$

uniformly in the range $y_0 \leqslant y \leqslant x^{1-\delta}$ and $y + y/\log^C y \leqslant z \leqslant y^2$. The constants implied by $\ll$ may depend on F, δ, C.

Combining Theorem 1 with (1.1), we see that

Corollary 2. Let $F(t) \in \mathbb{Z}[t]$ be irreducible. Fix $\delta > 0$. There is a constant $y_0 = y_0(\delta, F)$ such that uniformly for $y_0 \leqslant y \leqslant x^{1-\delta}$, we have

$$H_F(x, y, 2y) \asymp \frac{x}{(\log y)^{\mathcal{E}}(\log\log y)^{3/2}}$$

According the the above heuristic, it is natural to conjecture that the conclusion of Corollary 2 holds in a larger range of y, perhaps $y \leqslant x^{g-\delta}$. In particular, taking $y = x/2$, we conjecture that when $g \geqslant 2$, $H_F(x, x/2, x)$ has order $\frac{x}{(\log y)^{\mathcal{E}}(\log\log y)^{3/2}}$. If true, this is a large improvement over Tenenbaum's bound (1.2).

To prove Theorem 1, we develop a hybrid of the methods from [**27**] and [**7**]. The proof of the lower bound is accomplished in Section 3, and Section 4 contains the proof of the upper bound. A crucial device used in the upper bound in [**7**] is not available in the context of divisors of polynomials, and we must develop an alternative approach.

We note the formula

$$H_F(x, y, z) = \sum_{k=1}^{\infty}(-1)^{k-1} \sum_{y<d_1<\cdots<d_k\leqslant z} \Big(\Big\lfloor \frac{x}{[d_1, ..., d_k]}\Big\rfloor \rho(\mathrm{lcm}[d_1, ..., d_k]) + O\big(\rho(\mathrm{lcm}[d_1, ..., d_k])\big)\Big),$$

a consequence of inclusion-exclusion. However, this has too many summands to be of any use in bounding $H_F(x, y, z)$ unless the interval $(y, z]$ is very short.

2. Preliminaries

2.1. **Notation.** The symbols p, q (with or without subscripts) always denote primes. Constants implied by $O, \ll, \gg$ and $\asymp$ symbols depend on F, δ and C in Theorem 1. Dependence on any other parameter will be indicated, e.g. by a subscript. The notation $f \asymp g$ means $f \ll g$ and $g \ll f$.

The symbol p, with or without subscripts, always denotes a prime. Let $P^+(n)$ be the largest prime factor of n, and $P^-(n)$ be the smallest prime factor of n. Adopt the conventions $P^+(1) = 0$ and $P^-(1) = \infty$. For any $t \geqslant s \geqslant 1$, denote by $\mathscr{P}(s, t)$ the set of squarefree positive integers composed only of prime factors $p \in (s, t]$. In particular, $1 \in \mathscr{P}(s, t)$ for any s, t. Let

$$\tau(n; y, z) = \#\{d|n : y < d \leqslant z\}.$$

Given an integer $n \geqslant 1$, we say $d|n^\infty$ if every prime factor of d divides n. As noted earlier, we denote by $\rho(d)$ the number of solutions of the congruence

$$F(n) \equiv 0 \pmod{d}.$$

It follows from the Chinese remainder theorem that $\rho(n)$ is a multiplicative function of n. Let D_F be the discriminant of $F(X)$. Then we have (cf. Theorems 42, 52, 54 of Nagell [**25**][1]) for any prime p and positive integer a,

$$\rho(p^a) \leqslant \begin{cases} g & \text{if } p \nmid D_F, \\ gD_F^2 & \text{if } p | D_F. \end{cases} \tag{2.1}$$

[1]Nagell uses the term "primitive" to refer to a polynomial with the greatest common divisor of its coefficients equal to 1.

We also associate with F an Euler-like function

$$\varphi_F(n) := n\prod_{p|n}(1-\rho(p)/p). \tag{2.2}$$

In particular, we have $\varphi_F(n)\neq 0$ whenever $P^-(n) > gD_F^2$.

As in [**7**], for a given pair (y,z) with $4\leqslant y<z$, we define η,u,β,ξ by

$$z=e^\eta y = y^{1+u},\quad \eta=(\log y)^{-\beta},\quad \beta=\log 4-1+\frac{\xi}{\sqrt{\log\log y}}. \tag{2.3}$$

For $z\leqslant ey$, we need the following function

$$G(\beta)=\begin{cases}\beta, & \text{if } \beta\geqslant \log 4-1,\\ \frac{1+\beta}{\log 2}\log\left(\frac{1+\beta}{e\log 2}\right)+1, & \text{if } 0\leqslant\beta\leqslant\log 4-1,\end{cases} \tag{2.4}$$

as well as

$$z=z_0(y):=y\exp\{(\log y)^{1-\log 4}\}\approx y+y/(\log y)^{\log 4-1}. \tag{2.5}$$

With this notation, given any $\delta>0$, we have [**7**, Theorem 1], uniformly for $3\leqslant y\leqslant x^{1-\delta}$,

$$\frac{H(x,y,z)}{x}\asymp\begin{cases}\log(z/y)=\eta & y+1\leqslant z\leqslant z_0(y)\\ \dfrac{\beta}{\max(1,-\xi)(\log y)^{G(\beta)}} & z_0(y)\leqslant z\leqslant 2y\\ u^\delta(\log\frac{2}{u})^{-3/2} & 2y\leqslant z\leqslant y^2\\ 1 & z\geqslant y^2.\end{cases}$$

Our goal is to show the same bounds for $H_F(x,y,z)$

2.2. **Bakground lemmata.** Our first result is a consequence of the Prime Ideal Theorem with classical de la Valée Poussin error term (see [**20**], Satz 190).

Lemma 2.1. *There are two positive constants c_1 and c_2, which depend on F, such that*

$$\sum_{p\leqslant x}\frac{\rho(p)}{p}=\log\log x+c_1+O(e^{-c_2\sqrt{\log x}}).$$

In [**3**], Erdős showed that $\sum_{n=1}^x\rho(n)>cx$ for some constant c when x is sufficiently large. This was sharpened by Fomenko [**6**] and Kim [**16**], the sharpest known bounds (for large degree g) being the result of Lü [**21**]. We shall only require a very weak version of the bound.

Lemma 2.2 ([**21**, Theorems 1.1,1.2])**.** *For any $\varepsilon>0$, we have*

$$\sum_{d\leqslant x}\rho(d)=A_Fx+O_\varepsilon(x^{1-3/(g+6)+\varepsilon})$$

where A_F is a constant depending on F.

We need a generalization of a bound from Tenenbaum [**27**].

Lemma 2.3 (Tenenbaum [**27**, Lemma 3.4])**.** *Suppose that Q is an integer divisible by D_F. Let d_0, d_1 be two integers such that $d_0|Q^\infty$, $(d_1, Q) = 1$. Let K, K' be real number satisfying $0 < K < 1 \leqslant K'$. Suppose that x is sufficiently large, depending only on K, K'. Then there is a positive constant $c_3 = c_3(K, F, Q) < 1$ such that under the conditions*

$$2 \leqslant t \leqslant x^{c_3}, \quad d_0 d_1 \leqslant x^{1-K}, \quad P^+(d_0 d_1) \leqslant t^{K'},$$

we have

$$\sum_{\substack{n \leqslant x, d_0 d_1 | F(n) \\ p|F(n) \Rightarrow p|Qd_1 \text{ or } p>t}} 1 \asymp_{K,K',Q} \frac{x}{\log t} \frac{\rho(d_0)}{d_0} \frac{\rho(d_1)}{\varphi_F(d_1)}. \tag{2.6}$$

Moreover, the same order lower bound follows when n is restricted to $(x/2, x]$. In addition, the relation (2.6) holds, replacing the sign $\asymp_{K,K'}$ with $\ll_{K,K',F}$, when $P^+(d_0 d_1) > t^{K'}$ or $x^{c_3} < t \leqslant x$.

Proof. This follows from the proof of Tenenabum [**27**, Lemma 3.4]]; there, the lemma is proved when $Q = D_F F(1)$ and counting all $n \leqslant x$, but the same proof works for $x/2 < n \leqslant x$ and an arbitrary Q divisible by D_F. Also, the case $x^{c_3} < t \leqslant x$ is not considered explicitly in [**27**]. However, the stated result follows by applying [**27**, Lemma 3.4] with t replaced by $t' := \min(x^{c_3}, t)$, and noting that $\log t' \asymp_{K,F,Q} \log t$ when $t \leqslant x$. □

Lemma 2.4 (Tenenbaum [**27**, Lemma 3.7])**.** *We have uniformly for $w \geqslant v \geqslant 2$, $x \geqslant 2$ that*

$$\left| \left\{ n \leqslant x : \prod_{\substack{p^a \| F(n) \\ p \leqslant v}} p^a > w \right\} \right| \ll x \exp\left\{ -c_4 \frac{\log w}{\log v} \right\},$$

where $c_4 = c_4(F)$ is a positive constant.

Lemma 2.5 (Norton [**26**, §4])**.** *Suppose $0 \leqslant h < m \leqslant x$ and $m - h \geqslant \sqrt{x}$. Then*

$$\sum_{h \leqslant k \leqslant m} \frac{x^k}{k!} \asymp \min\left(\sqrt{x}, \frac{x}{x-m}\right) \frac{x^m}{m!}.$$

To understand the global distribution of the divisors of integers, we introduce a function which measures the degree of clustering of the divisors of an integer a. For $\sigma > 0$, we define

$$\mathscr{L}(a;\sigma) = \{x \in \mathbb{R} : \tau(a; e^x, e^{x+\sigma}) \geqslant 1\}$$

and

$$L(a;\sigma) = \text{meas}\mathscr{L}(a;\sigma),$$

where $\text{meas}(\cdot)$ denotes Lebesgue measure. We record easy bounds for $L(a;\sigma)$.

Lemma 2.6. *We have*

(i) *If* $(a,b)=1$*, then* $L(ab;\sigma)\leqslant\tau(b)L(a;\sigma)$*;*

(ii) *If* $p_1<\cdots<p_k$*, then*

$$L(p_1\cdots p_k;\sigma)\leqslant\min_{0\leqslant j\leqslant k}2^{k-j}(\log(p_1\cdots p_j)+\sigma).$$

(iii) *For any* $a\in\mathbb{N}$ *and* $\sigma>0$ *we have*

$$L(a;\sigma)\geqslant\sigma(2\tau(a)-W(a;\sigma)),$$

where

$$W(a;\sigma)=|\{d|a,d'|a:|\log(d/d')|\leqslant\sigma\}|. \tag{2.7}$$

Proof. Parts (i) and (ii) are proved in Ford [**7**],Lemma 3.1. To show part (iii), let $\mathcal{D}$ be the set of divisors $d|a$ such that there is no divisor $d'|a$ with $|\log(d/d')|\leqslant\sigma$ (isolated divisors). The desired inequality follows from the fact that $L(a;\sigma)\geqslant\sigma|\mathcal{D}|$ and

$$W(a;\sigma)\geqslant\tau(a)+(\tau(a)-|\mathcal{D}|).$$

□

Lemma 2.7. *For any* $r<s<t$ *and* $\eta>0$ *we have*

$$\sum_{a\in\mathscr{P}(r,t)}\frac{L(a;\eta)\rho(a)}{a}\ll\left(\frac{\log t}{\log s}\right)^2\sum_{a\in\mathscr{P}(r,s)}\frac{L(a;\eta)\rho(a)}{a}.$$

Proof. For any $a\in\mathscr{P}(r,t)$, decompose a uniquely as $a=a'a''$ where $P^+(a')\leqslant s<P^-(a'')$, and write $L(a;\eta)\leqslant\tau(a'')L(a';\eta)$ from Lemma 2.6 (i). Using Lemma 2.1 we have

$$\sum_{a''\in\mathscr{P}(s,t)}\frac{\tau(a'')\rho(a'')}{a''}=\prod_{s<p\leqslant t}\left(1+\frac{2\rho(p)}{p}\right)\ll\left(\frac{\log t}{\log s}\right)^2$$

and the proof is complete. □

3. LOWER BOUND

In this section we prove the lower bound in Theorem 1. As in [**7**], we first bound $H_F(x,y,z)$ in terms of an average of $L(a;\eta)\rho(a)/a$. This be thought of as a kind of local-to-global principle. Recall the definition (2.3) of η. Also define

$$D=10gD_F^2,\qquad Q=\prod_{p\leqslant D}p. \tag{3.1}$$

By (2.1), we have

$$\varphi_F(p^a)\geqslant\frac{p^a}{2}\qquad(p\nmid Q). \tag{3.2}$$

Proposition 3.1. *Let C and δ be two positive real numbers with $0 < \delta < 1$. Suppose that y is sufficiently large (depending on F, δ, C), $y < z = e^{\eta} y \leqslant x^{1-\delta/2}$, and $\frac{1}{\log^C y} \leqslant \eta \leqslant \log y$ (in particular, $z \leqslant y^2$). Then*

$$H_F(x, y, z) - H_F(x/2, y, z) \gg \frac{x}{\log^2 y} \sum_{a \in \mathscr{P}(D,z)} \frac{L(a;\eta)\rho(a)}{a}. \tag{3.3}$$

Proof. Define

$$\nu = \min\left\{ c_3\left(\frac{\delta}{3}, F, Q\right), \frac{\delta}{3} \right\}, \qquad \varepsilon = \frac{\nu}{6g},$$

with $c_3(\frac{\delta}{3}, F, Q)$ being defined as in Lemma 2.3. Let

$$\mathcal{A} = \{a \in \mathbb{N} : a \leqslant y^{\nu}, \rho(a) > 0, \mu^2(a) = 1, (a, Q) = 1\}.$$

For $a \in \mathcal{A}$, we consider integers $n \in (x/2, x]$ such that $F(n)$ has the decomposition

$$F(n) = apb \tag{3.4}$$

satisfying the following conditions

$$(*)\left\{\begin{array}{l} \text{(i) } p \text{ is a prime factor of } F(n) \text{ with } p > D \text{ and } \log(\frac{y}{p}) \in \mathscr{L}(a;\eta), \\ \text{(ii) Every prime factor } q|b \text{ satisfies } q|apQ \text{ or } q > R := \min(z, x^{\nu}). \end{array}\right\}$$

If $F(n)$ satisfies $(*)$, then there is a divisor d of a such that $y < pd \leqslant z$, which implies that

$$d \leqslant a \leqslant y^{\nu} < y^{1-\nu} \leqslant \frac{y}{d} < p \leqslant z. \tag{3.5}$$

In particular, $p > a$ implies that $(a, p) = 1$.

With n fixed, let $r(n)$ be the number of triples a, p, b such that (3.4) holds subject to $(*)$. We assume that y is large enough so that $y^{\nu} > Q^2$. Thus, (3.5) imply that $p > D$. We claim that $r(n) \ll 1$ for all n. If $z \leqslant x^{\nu}$, then $R = z$ and it is clear from (3.5) that a, p, b are unique. Hence $r(n) \leqslant 1$. If $z > x^{\nu}$, then $R = x^{\nu}$ and $y \geqslant z^{1/2} \geqslant x^{\nu/2}$. Since $F(n) \ll x^g$ for $n \leqslant x$, we see that $F(n)$ has $O(1)$ prime factors (counted with multiplicity) larger than y^{ν}. By (3.5), a must contain all of the prime factors of $F(n)$ which are below y^{ν}, except for those dividing Q. There are $O(1)$ possible ways of distributing the prime factors of $F(n)$ which are $> y^{\nu}$ among the numbers b and p, and therefore $r(n) \ll 1$ in this case.

Therefore, we have

$$\begin{aligned} H_F(x, y, z) - H_F(x/2, y, z) &\geqslant \sum_{\substack{x/2<n\leqslant x \\ r(n)>0}} 1 \gg \sum_{x/2<n\leqslant x} r(n) \\ &\geqslant \sum_{a\in\mathcal{A}} \sum_{\log(y/p)\in\mathscr{L}(a;\eta)} \sum_{\substack{x/2<n\leqslant x, ap|F(n) \\ q|F(n) \Rightarrow q|apQ \text{ or } q>R}} 1. \end{aligned} \tag{3.6}$$

Moreover,

$$ap \leqslant az \leqslant z^{1+\nu} \leqslant x^{(1-\delta/2)(1+\nu)} \leqslant x^{(1-\delta/2)(1+\delta/3)} < x^{1-\delta/6},$$

as well as

$$R \leqslant x^{c_3(\frac{\delta}{3},F,Q)}, \qquad P^+(pa) \leqslant z \leqslant R^{1/\nu}.$$

Applying Lemma 2.3 with $d_0 = 1$, $d_1 = ap$, $K = \delta/3$ and $K' = 1/\nu$, we find that

$$\sum_{\substack{x/2<n\leqslant x, ap|F(n) \\ q|F(n)\Rightarrow q|apQ \text{ or } q>R}} 1 \gg \frac{x}{\log R}\frac{\rho(ap)}{\varphi_F(ap)} \gg \frac{x}{\log R}\frac{\rho(a)}{a}\frac{\rho(p)}{p}.$$

Thus, from (3.6), we derive that

$$H_F(x,y,z) - H_F(x/2,y,z) \gg \frac{x}{\log y}\sum_{a\in\mathcal{A}}\frac{\rho(a)}{a}\sum_{\log(\frac{y}{p})\in\mathscr{L}(a;\eta)}\frac{\rho(p)}{p}.$$

Now $\mathscr{L}(a;\eta)$ is the disjoint union of intervals of length between $\eta/2$ and η, and $\eta \gg 1/(\log y)^C$ by assumption. Hence, using $p > y^{1-\nu}$, repeated application of Lemma 2.1 implies

$$\sum_{\log(y/p)\in\mathscr{L}(a;\eta)}\frac{\rho(p)}{p} \gg \frac{L(a;\eta)}{\log y}.$$

We conclude that

$$H_F(x,y,z) - H_F(x/2,y,z) \gg \frac{x}{\log^2 y}\sum_{a\in\mathcal{A}}\frac{L(a;\eta)\rho(a)}{a}. \tag{3.7}$$

We next relax the condition $a \leqslant y^\nu$ in the summation over a. Recall that $\varepsilon = \nu/(6g)$. We have

$$\sum_{\substack{a\leqslant y^\nu \\ (a,Q)=1 \\ \mu^2(a)=1}}\frac{L(a;\eta)\rho(a)}{a} \geqslant \sum_{a\in\mathscr{P}(D,y^\varepsilon)}\frac{L(a;\eta)\rho(a)}{a}\left(1-\frac{\log a}{\log(y^\nu)}\right). \tag{3.8}$$

Write $\log a = \sum_{p|a}\log p$, $a = pf$ with $(p,f) = 1$, use $\rho(fp) = \rho(p)\rho(f) \leqslant g\rho(f)$ by (2.1) and $L(pf;\eta) \leqslant 2L(f;\eta)$ from Lemma 2.6 (i). This gives

$$\begin{aligned}\sum_{a\in\mathscr{P}(D,y^\varepsilon)}\frac{L(a;\eta)\rho(a)\log a}{a} &\leqslant 2g\sum_{D<p\leqslant y^\varepsilon}\frac{\log p}{p}\sum_{f\in\mathscr{P}(D,y^\varepsilon)}\frac{L(f;\eta)\rho(f)}{f} \\ &\leqslant 2g(\log(y^\varepsilon)+O(1))\sum_{f\in\mathscr{P}(D,y^\varepsilon)}\frac{L(f;\eta)\rho(f)}{f},\end{aligned}$$

by Mertens' estimate. If y is sufficiently large in terms of ε and F, then $2g(\log(y^\varepsilon)+O(1)) \leqslant \frac{\nu}{2}\log y$. Inserting this last bound into (3.8), we obtain

$$\sum_{a\in\mathcal{A}}\frac{L(a;\eta)\rho(a)}{a} = \sum_{\substack{a\leqslant y^\nu \\ (a,Q)=1 \\ \mu^2(a)=1}}\frac{L(a;\eta)\rho(a)}{a} \geqslant \frac{1}{2}\sum_{a\in\mathscr{P}(D,y^\varepsilon)}\frac{L(a;\eta)\rho(a)}{a}.$$

Inserting this into (3.7), and applying Lemma 2.7 with $t = z$ and $s = y^\varepsilon$, we conclude the proof. □

Next, as in [**7**], we relate the sum over a in Lemma 3.1 to an average of the function $W(a;\eta)$ from (2.7).

Lemma 3.2. *Let C and δ be two positive real numbers with $0 < \delta < 1$. Suppose y is sufficiently large (depending on F, C, δ), $y < z = e^\eta y \leqslant x^{1-\delta/2}$, and $\frac{1}{\log^C y} \leqslant \eta \leqslant \log y$ (in particular, $z \leqslant y^2$). Then*

$$H_F(x,y,z) - H_F(x/2,y,z) \gg \frac{\eta(1+\eta)x}{\log^2 y} \sum_{a\in\mathscr{P}(\max(D,z/y),z)} \frac{(2\tau(a) - W(a;\eta))\rho(a)}{a}.$$

Proof. In the summation on the right side of (3.3), decompose a uniquely as $a = a'a''$, where $P^+(a') \leqslant z/y < P^-(a'')$. As in [**7**, Lemma 4.2], the prime factors $\leqslant z/y$ have little effect, and we do not lose much using the trivial inequality

$$L(a;\eta) \geqslant L(a'';\eta).$$

Therefore, because ρ is multiplicative,

$$\sum_{a\in\mathscr{P}(D,y^\varepsilon)} \frac{L(a;\eta)\rho(a)}{a} \geqslant \sum_{a'\in\mathscr{P}(D,z/y)} \frac{\rho(a')}{a'} \sum_{a''\in\mathscr{P}(\max(D,z/y),z)} \frac{L(a'';\eta)\rho(a'')}{a''}.$$

Writing the sum on a' as an Euler product, and then using Lemma 2.1 we see that

$$\sum_{a'\in\mathscr{P}(D,z/y)} \frac{\rho(a')}{a'} = \prod_{D<p\leqslant z/y} \left(1 + \frac{\rho(p)}{p}\right) \gg \exp\left\{\sum_{D<p\leqslant z/y} \frac{\rho(p)}{p}\right\} \gg 1 + \log(z/y) = 1 + \eta.$$

Finally, in the sum over a'', we invoke Lemma 2.6 (iii) to obtain $L(a'';\eta) \geqslant \eta(2\tau(a'') - W(\alpha'';\eta))$, and the proof is complete. □

From Lemma 3.2, to obtain a lower bound for $H_F(x,y,z)$, we need to provide an upper bound on the sum over $\frac{W(a;\eta)\rho(a)}{a}$. For the purpose, we partition the primes into sets $E_1, E_2, \ldots$ and then consider those integers a with a prescribed number of prime factors in each interval E_j. The partition is similar to that in [**7**, Section 4]. Each E_j consists of the primes in an interval $(\lambda_{j-1}, \lambda_j]$, where $\lambda_0 = D$ and $\lambda_j \approx \lambda_{j-1}^2$; specifically, λ_j is defined inductively for $j \geqslant 1$ as the largest prime so that

$$\sum_{\lambda_{j-1}<p\leqslant\lambda_j} \frac{\rho(p)}{p} \leqslant \log 2. \tag{3.9}$$

Note that $\rho(p)/p \leqslant \log 2$ always holds when $p > \lambda_0 = D$, so that each set E_j is nonempty.

By Lemma 2.1, we have

$$\log\log\lambda_j - \log\log\lambda_{j-1} = \log 2 + O(e^{-c_2\sqrt{\log\lambda_{j-1}}}),$$

By summing the above sum from $r = 1$ to j, we get

$$\log \lambda_j - \log(D) = j \log 2 + O\Big(\sum_{r=1}^{j} e^{-c_2\sqrt{\log \lambda_{r-1}}} \Big) = j \log 2 + O(1),$$

which implies that

$$2^{j-c_5} \leqslant \log \lambda_j \leqslant 2^{j+c_5} \quad (j \geqslant 0) \tag{3.10}$$

for some absolute constant c_5. For a vector $\mathbf{b} = (b_1, ..., b_J)$ of non-negative integers, let $\mathcal{A}(b)$ be the set of square-free integers a composed of exactly b_j prime factors from E_j for each j. The following is analogous to [**7**, Lemma 4.7]. Here M is a sufficiently large constant, which depends only on F, and hence M depend on c_1, c_2, c_4, c_5 as well.

Lemma 3.3. *Suppose $\eta > 0$, $\mathbf{b} = (b_1, ..., b_h)$ and define $m = \min\{j : b_j \geqslant 1\}$. If $\eta < 1$, further assume that $m \geqslant M$ and that $b_j \leqslant 2^{j/2}$ for each j. Then*

$$\sum_{a \in \mathcal{A}(\mathbf{b})} \frac{W(a;\eta)\rho(a)}{a} \leqslant \frac{(2\log 2)^{b_m+\cdots+b_h}}{b_m! \cdots b_h!} \Big[1.01 + (2^{c_5} c_6 g)\eta \sum_{j=m}^{h} 2^{-j+b_m+\cdots b_j} \Big) \Big].$$

for some absolute constant $c_6 > 0$.

Proof. Let $k = b_m + \cdots + b_h$. For $j \geqslant 0$, let $k_j = \sum_{i \leqslant j} b_i$. Let $a = p_1 \cdots p_k$, where

$$p_{k_{j-1}+1}, \cdots, p_{k_j} \in E_j \quad (m \leqslant j \leqslant h) \tag{3.11}$$

and the primes in each interval E_j are unordered. Since $W(p_1 \cdots p_k; \eta)$ is the number of pairs $Y, Z \subseteq \{1, \cdots, k\}$ with

$$-\eta \leqslant \sum_{i \in Y} \log p_i - \sum_{i \in Z} \log p_i \leqslant \eta, \tag{3.12}$$

we have

$$\sum_{a \in \mathcal{A}(\mathbf{b})} \frac{W(a;\eta)\rho(a)}{a} \leqslant \frac{1}{b_m! \cdots b_h!} \sum_{Y,Z \subseteq \{1,\ldots,k\}} \sum_{\substack{p_1,\ldots,p_k \\ (3.11),(3.12)}} \frac{\rho(p_1 \cdots p_k)}{p_1 \cdots p_k}. \tag{3.13}$$

There are 2^k pairs (Y, Z) with $Y = Z$, and thus these pairs contribute at most $\frac{(2\log 2)^k}{b_m! \cdots b_h!}$ to the right side of (3.13).

When $Y \neq Z$, let $I = \max(Y \Delta Z)$ and we will split off the term $\log p_I$ from the inequalities (3.12). Define $E(I)$ by $k_{E(I)-1} < I \leqslant k_{E(I)}$, so that $p_I \in E_{E(I)}$. Let

$$\ell = \min\{j : \lambda_j \geqslant \eta^{-2}\}. \tag{3.14}$$

We distinguish two cases: (i) $E(I) > \ell$; (ii) $m \leqslant E(I) \leqslant \ell$.

Consider first a pair Y, Z in case (i). With p_i all fixed for $i \neq I$, (3.12) implies that p_I lies in an interval of the form $(U, Ue^{2\eta}]$, where $U \geqslant \lambda_{E(I)} \geqslant \eta^{-2}$ depends on p_i for $i \neq I$. By (2.1),

$$\sum_{U < p_I \leqslant e^{2\eta} U} \frac{\rho(p_I)}{p_I} \leqslant g \sum_{U < p_I \leqslant e^{2\eta} U} \frac{1}{p_I} \leqslant c_6 g \frac{\eta}{\log U} \leqslant c_6 g \eta 2^{-E(I)+c_5}$$

for an absolute constant c_6 (here we use the Brun-Titchmarsh inequality for $\eta < 1$ and the Mertens' bound for primes when $\eta \geqslant 1$). Therefore, with Y and Z fixed, the sum over $p_1, \dots, p_k$ on the right side of (3.13) is at most

$$c_6 g \eta 2^{-E(I)+c_5} (\log 2)^{k-1},$$

using (3.9). With I fixed there are 2^{k-1+I} pairs Y, Z. We also have

$$\sum_{I=1}^{k} 2^{I-E(I)} = \sum_{j=m}^{h} 2^{-j} \sum_{k_{j-1} < I \leqslant k_j} 2^I \leqslant 2 \sum_{j=m}^{h} 2^{-j+k_j}.$$

We find that the contribution to the right side of (3.13) from those Y, Z counted in case (i) is

$$\leqslant (2^{1+c_5} c_6 g) \frac{\eta (2\log 2)^k}{b_m! \cdots b_h!} \sum_{j=m}^{h} 2^{-j+b_m+\cdots b_j}.$$

In case (ii), (3.14) implies

$$\eta \leqslant \lambda_{\ell-1}^{-1/2} \leqslant \exp\{-2^{\ell-1-c_5}\}. \tag{3.15}$$

Write

$$a = a' p_{k_\ell+1} \cdots p_k, \quad a' = p_1 \cdots p_{k_\ell}.$$

By hypothesis, $Y \bigcap \{k_\ell + 1, ..., k\} = Z \bigcap \{k_\ell + 1, ..., k\}$. We use a trivial bound (3.9) for the sums over $p_{k_\ell+1}, \dots, p_k$ on the right side of (3.13), summing over the 2^{k-k_ℓ} possibilities for the set $Y \bigcap \{k_\ell + 1, ..., k\} = Z \bigcap \{k_\ell + 1, ..., k\}$, then expressing the remaining sum over $p_1, \dots, p_{k_\ell}$, $Y \cap \{1, \dots, k_\ell\}$ and $Z \cap \{1, \dots, k_\ell\}$ in terms of a sum on a'. We conclude that the contribution to the right side of (3.13) from those Y, Z counted in case (ii) is

$$\leqslant \frac{(2\log 2)^{k-k_\ell}}{b_{\ell+1}! \cdots b_h!} \sum_{a'} \frac{(W(a';\eta) - \tau(a'))\rho(a')}{a'}. \tag{3.16}$$

The factor $W(a';\eta) - \tau(a')$ arises due to our counting only of sets with $Y \neq Z$. From (2.7), we see that

$$W(a';\eta) - \tau(a') = 2\#\{(d_1, d_2) : d_1 | a', d_2 | a', 1 < d_2/d_1 \leqslant e^\eta\}.$$

Suppose $d_1|a', d_2|a'$ and $1 < d_2/d_1 \leqslant e^\eta$. Let $d = (d_1, d_2), d_1 = f_1 d, d_2 = f_2 d$ and $a' = df_1 f_2 a''$. Since $\rho(f_2) \leqslant g^{\omega(f_2)} \leqslant g^{k_\ell}$ by (2.1), we obtain

$$\begin{aligned}\sum_{a'} \frac{(W(a';\eta) - \tau(a'))\rho(a')}{a'} &\leqslant 2 \sum_{a''df_1 \in \mathscr{P}(\lambda_0,\lambda_\ell)} \frac{\rho(a''df_1)}{a''df_1} \sum_{f_1 < f_2 \leqslant e^\eta f_1} \frac{\rho(f_2)}{f_2} \\ &\leqslant 2g^{k_\ell} \sum_{a''df_1 \in \mathscr{P}(\lambda_0,\lambda_\ell)} \frac{\rho(a''df_1)}{a''df_1} \sum_{f_1 < f_2 \leqslant e^\eta f_1} \frac{1}{f_2} \\ &\leqslant 4g^{k_\ell}\eta \sum_{a''df_1 \in \mathscr{P}(\lambda_0,\lambda_\ell)} \frac{\rho(a''df_1)}{a''df_1} \\ &\leqslant 4g^{k_\ell}\eta \prod_{\lambda_0 < p \leqslant \lambda_\ell} \Big(1 + \frac{\rho(p)}{p}\Big)^3 \\ &\leqslant 4g^{k_\ell}\eta \exp\Big\{3 \sum_{\lambda_0 < p \leqslant \lambda_\ell} \frac{\rho(p)}{p}\Big\} \\ &\leqslant g^{k_\ell} 2^{3\ell+2}\eta,\end{aligned}$$

where we used (3.9) in the last step. Inserting this last bound into (3.16), we see that the contribution to the right side of (3.13) from the sets Y, Z in case (ii) is at most

$$\frac{(2\log 2)^k}{b_m! \cdots b_h!} V, \qquad V = g^{k_\ell} 2^{3\ell+2} b_m! \cdots b_\ell! \eta.$$

By assumption, $k_\ell \leqslant 4 \cdot 2^{\ell/2}$. Using (3.15) and the bound $b_j \leqslant 2^{j/2}$, we see that

$$V \leqslant g^{2^{\ell/2+2}} 2^{3\ell+2} (2^{\lceil \ell/2 \rceil}!)^\ell \exp\{-2^{\ell-1-c_5}\} \leqslant 0.01$$

if M is large enough, depending on F (recall that $\ell \geqslant m \geqslant M$).

Combining the contributions from the case $Y = Z$ and $Y \neq Z$, we immediately get the required result. □

We now stitch together the contribution from many sets $\mathcal{A}(\mathbf{b})$, analogous to Lemma 4.8 in [**7**]. The proof is nearly identical, and so we only sketch it, indicating changes from [**7**].

Lemma 3.4. *Suppose* $y \geqslant y_0 = y_0(F, \delta, C)$ *and* $0 < \eta \leqslant 2^{-M-1} \log z$. *Define*

$$v = \Big\lfloor \frac{\log\log(z) - \max(0, \log \eta)}{\log 2} - M + 1 \Big\rfloor, \tag{3.17}$$

$$s = M + \max\Big(0, \Big\lfloor \frac{\log \eta}{\log 2} \Big\rfloor\Big) - \frac{\log(c_6 g \eta)}{\log 2} - c_5 - 5, \tag{3.18}$$

If

$$\max(10M, v) \leqslant k \leqslant \min\big(v + s - M/3 - 1, 100(v-1)\big). \tag{3.19}$$

then

$$\sum_{\substack{a\in\mathscr{P}(\max(D,e^\eta),z)\\ \omega(a)=k}} \frac{(2\tau(a)-W(a;\eta))\rho(a)}{a} \gg \frac{(2v\log 2)^k(k-v+1)}{(k+1)!}.$$

Proof. Let $m = M + \max\left(0, \lfloor\frac{\log\eta}{\log 2}\rfloor\right)$, put $h = v+m-1$ and define $\mathscr{B}$ to be the set of vectors $\mathbf{b} = (b_m, \ldots, b_h)$ satisfying

(a) $b_j = 0$ $(1 \leqslant j \leqslant m-1)$;
(b) $b_m + \cdots + b_h = k$; and
(c) $b_j \leqslant M + (j+1-M)^2$ for all $j \geqslant m$.

We assume that $M \geqslant c_5 + 1$, which ensures, by (3.10), that $P^-(a) > \lambda_{m-1} > e^\eta$ whenever $a \in \mathscr{A}(\mathbf{b})$ and $\mathbf{b} \in \mathscr{B}$. We also have $h \leqslant \frac{\log\log z}{\log 2} - c_5$, and thus for such a we have also $P^+(a) \leqslant \lambda_h \leqslant z$. That is,

$$\bigcup_{\mathbf{b}\in\mathscr{B}} \mathscr{A}(\mathbf{b}) \subset \mathscr{P}(\max(D,e^\eta),z).$$

By the definition of the sets E_j, for any $\mathbf{b} \in \mathscr{B}$, we have

$$\begin{aligned}
\sum_{a\in\mathscr{A}(\mathbf{b})} \frac{\rho(a)}{a} &= \prod_{j=m}^{h} \frac{1}{b_j!}\Big(\sum_{p_1\in E_j} \frac{\rho(p_1)}{p_1} \sum_{\substack{p_2\in E_j\\ p_2\neq p_1}} \frac{\rho(p_2)}{p_2} \cdots \sum_{\substack{p_{b_j}\in E_j\\ p_{b_j}\notin\{p_1,\ldots,p_{b_j-1}\}}} \frac{\rho(p_{b_j})}{p_{b_j}}\Big) \\
&\geqslant \prod_{j=m}^{h} \frac{1}{b_j!}\Big(\sum_{p\in E_j}\frac{\rho(p)}{p} - \frac{b_j-1}{\lambda_{j-1}}\Big)^{b_j} \\
&\geqslant \prod_{j=m}^{h} \frac{1}{b_j!}\Big(\log 2 - \frac{b_j}{\lambda_{j-1}}\Big)^{b_j} \\
&\geqslant \frac{(\log 2)^k}{b_m!\cdots b_h!} \prod_{j=m}^{h}\Big(1 - \frac{2^{j/10}}{\exp\{2^{j-1+c_3-c_4}\}}\Big)^{2^{j/10}} \\
&\geqslant 0.999\frac{(\log 2)^k}{b_m!\cdots b_h!}
\end{aligned} \tag{3.20}$$

provided M is large enough (recall $j \geqslant m \geqslant M$). combining this with Lemma 3.3 and (3.20), we see that

$$\sum_{a\in\mathscr{A}(\mathbf{b})} \frac{(2\tau(a)-W(a;\eta))\rho(a)}{a} \geqslant \frac{(2\log 2)^k}{b_m!\cdots b_h!}\Big[0.9 - (2^{c_5}c_6 g)\eta \sum_{j=m}^{h} 2^{-j+b_m+\cdots+b_j}\Big]. \tag{3.21}$$

For $1 \leqslant i \leqslant v$, set $g_i = b_{m-1+i}$. Let $\mathscr{G}$ denote the set of vectors $\mathbf{g} = (g_1, \ldots, g_v)$ such that

(d) $g_1 + \cdots + g_v = k$;
(e) $g_i \leqslant M + i^2$ for all i;

(f) $2^{m-1}\sum_{j=m}^{h} 2^{-j+b_m+\cdots+b_j} = \sum_{i=1}^{v} 2^{-i+g_1+\cdots+g_i} \leqslant 2^{s+1}$.

Clearly, (d) implies (b). Since $m \geqslant M$, item (e) implies (c). That is, $\mathbf{g} \in \mathscr{G}$ implies that $\mathbf{b} \in \mathscr{B}$. From the definition of s and the inequality in (f), we have $(2^{c_5}c_6 g)\eta 2^{s-m+2} \leqslant 2^{-3}$. By (3.21) and the equality in (f), we conclude that for all $\mathbf{g} \in \mathscr{G}$, and with $b_j = g_j - m + 1$ for each $j \geqslant m$,

$$\sum_{a\in\mathscr{A}(\mathbf{b})} \frac{(2\tau(a) - W(a;\eta))\rho(a)}{a} \geqslant \frac{(2\log 2)^k}{2g_1!\cdots g_v!}.$$

The argument on p. 418–419 of [**7**] then shows that

(3.22)
$$\sum_{\substack{a\in\mathscr{P}(\max(D,e^\eta),z)\\ \omega(a)=k}} \frac{(2\tau(a) - W(a;\eta))\rho(a)}{a} \geqslant (2\log 2)^k \sum_{\mathbf{g}\in\mathscr{G}} \frac{1}{g_1!\cdots g_v!} \geqslant (2v\log 2)^k \mathrm{Vol}\,\Gamma_k(s,v),$$

where $\Gamma_k(s,v)$ is the set of $\xi = (\xi_1, ..., \xi_k) \in \mathbb{R}^k$ satisfying

(i) $0 \leqslant \xi_1 \leqslant \cdots \leqslant \xi_k < 1$;

(ii) For $1 \leqslant i \leqslant \sqrt{k-M}$, $\xi_{M+i^2} > i/v$ and $\xi_{k+1-(M+i^2)} < 1 - i/v$;

(iii) $\sum_{j=1}^{k} 2^{j-v\xi_j} \leqslant 2^s$.

We note that our condition (e) is weaker than the corresponding condition in [**7**], thus the sum on the left side of (3.22) is greater than the sum considered in [**7**]. We easily verify that, if M is sufficiently large, then $s \geqslant M/2 + 1$. Thus, by (3.19), all of the hypotheses of [**7**, Lemma 4.9] are satisfied, and we conclude that

$$\mathrm{Vol}(\Gamma_k(s,v)) \gg \frac{k-v+1}{(k+1)!}.$$

Inserting this into (3.22), this completes the proof. □

Proof of the lower bounds in Theorem 1. Suppose $2 \leqslant y \leqslant x^{1-\delta}$ and $\frac{1}{\log^C y} \leqslant \eta \leqslant 1/100$, and define β, ξ by (2.3) and $G(\beta)$ by (2.4). Let $y \geqslant y_0(F, C, \delta)$. Define v and s by (3.17), (3.18), respectively. We will apply Lemma 3.4 for all k satisfying

(3.23)
$$\left(1 + \frac{\beta}{10B}\right)v \leqslant k \leqslant \min(1+\beta, \log 4)v.$$

This includes at least one value of k since $\frac{\log 100}{\log\log y} \leqslant \beta \leqslant B$. Also, by (2.3),

$$k - v \leqslant \beta v = \frac{-\log\eta}{\log\log y} v \leqslant s - M/3 - 1,$$

and we have that $v \geqslant 10M$ for large enough y_0. Hence, (3.19) holds for all k satisfying (3.23). For each such k in (3.23), we obtain

$$\sum_{\substack{a\in\mathscr{P}(\max(D,e^\eta),z)\\ \omega(a)=k}} \frac{(2\tau(a) - W(a;\eta))\rho(a)}{a} \gg \beta\frac{(2v\log 2)^k}{k!}$$

Applying Lemma 3.2 and using Lemma 2.5 to bound the resulting sum over k (cf., p. 397–398 in [**7**]), we see that

$$H_F(x,y,z) - H_F(x/2,y,z) \gg \frac{\beta\eta(1+\eta)x}{\log^2 y} \sum_{k:(3.23)} \frac{(2v\log 2)^k}{k!} \gg \frac{\beta x}{\max(1,-\xi)(\log y)^{G(\beta)}}.$$

This gives the lower bound in Theorem 1 when $\eta \leqslant \frac{1}{100}$.

Next, let $\gamma = 2^{-M-c_5}(c_6 g)^{-1}\delta$, which is smaller than $\delta/3$, and suppose that $\frac{1}{100} \leqslant \eta \leqslant \gamma \log y$. Apply Proposition 3.1, followed by Lemma 3.4 with the single term $k = v$. Recalling that $\eta = u \log y$, we conclude that

$$H_F(x,y,z) - H_F(x/2,y,z) \gg \frac{\eta^2 x}{\log^2 y} \frac{(2v\log 2)^v}{(v+1)!} \gg \frac{x u^{\mathcal{E}}}{(\log \frac{2}{u})^{3/2}},$$

as required for Theorem 1. □

Finally, when $y^{1+\gamma} \leqslant z \leqslant y^2$ we have trivially

$$H_F(x,y,z) - H_F(x/2,y,z) \geqslant H_F(x,y,y^{1+\gamma}) - H_F(x/2,y,y^{1+\gamma}) \gg x.$$

4. THE UPPER BOUND IN THEOREM 1, PART I

In this section, we establish the principal local-to-global result needed for the upper bound in Theorem 1. A crucial tool from [**7**] is, however, unavailable because if $g = \deg(F) \geqslant 2$, $n \asymp x$ and $d|F(n)$ with $y < d \leqslant z \leqslant x^{1-\delta}$, then the complementary divisor $F(n)/d$ is $\gg x^{g-1+\delta}$ and this is too large to handle. We get around this with another method (surrounding the parameters $A_{n,d}$, $B_{n,d}$ below). Recall that $\mathscr{P}(s,t)$ is the set of square-free integers, all of whose prime factors lie in $(s,t]$.

Proposition 4.1. *Let C and δ be two positive real numbers with $0 < \delta < 1 < C$. Suppose $y_0 = y_0(F,\delta,C)$ is sufficiently large. Then for $y_0 \leqslant y < z = e^\eta y \leqslant x^{1-\delta}$ and $\frac{1}{\log^C y} \leqslant \eta$, we have*

$$H_F(x,y,z) \ll \frac{x}{\log^2 y} \sum_{a\in\mathscr{P}(D,z)} \frac{L(a;\eta)\rho(a)}{\varphi_F(a)},$$

where D is defined in (3.1).

4.1. Reduction of complicated sums to simpler ones.

In this subsection, we present ways of bounding certain complicated sums by simpler ones. Our main result is similar in spirit to Lemma 3.3 of [**17**]. For all positive integers n with $n > \sqrt{X}$, we define

$$h(n;X) := \min\Big\{\text{prime } q : \prod_{p^\nu \| n, p\leqslant q} p^\nu > \sqrt{X}\Big\}. \tag{4.1}$$

Lemma 4.2. *Let $100 \leqslant X \leqslant z$. Then*

$$\sum_{\substack{\ell > \sqrt{X} \\ P^-(\ell) > D \\ P^+(\ell)\leqslant z}} \frac{L(\ell;\eta)\rho(\ell)}{\varphi_F(\ell)\log^2 h(\ell;X)} \ll \frac{1}{\log^2 X}\left(\frac{\log z}{\log X}\right)^{4g} \sum_{a\in\mathscr{P}(D,z)} \frac{L(a;\eta)\rho(a)}{\varphi_F(a)}. \tag{4.2}$$

In addition,

$$\sum_{\substack{P^-(\ell)>D\\ P^+(\ell)\leqslant z}} \frac{L(\ell;\eta)\rho(\ell)}{\varphi_F(\ell)\log^2(P^+(\ell)+z^{3/4}/\ell)} \ll \frac{1}{\log^2 z}\sum_{a\in\mathscr{P}(D,z)}\frac{L(a;\eta)\rho(a)}{\varphi_F(a)}. \tag{4.3}$$

Proof. Our first goal is to prove that

$$\sum_{\substack{\ell>\sqrt{X}\\ P^-(\ell)>D\\ P^+(\ell)\leqslant z}} \frac{L(\ell;\eta)\rho(\ell)}{\varphi_F(\ell)\log^2 h(\ell;X)} \ll \frac{1}{\log^2 X}\left(\frac{\log z}{\log X}\right)^{4g}\sum_{\substack{P^-(a)>D\\ P^+(a)\leqslant z}}\frac{L(a;\eta)\rho(a)}{\varphi_F(a)}. \tag{4.4}$$

Let $f(n)=\frac{L(n;\eta)\rho(n)}{\varphi_F(n)/n}$ for $(n,Q)=1$ and $f(n)=0$ for $(n,Q)>1$. By Lemma 2.6 (i) and the fact that $D>2g$ from (3.1), we see that $f(p^\nu m)\leqslant (4g)^\nu f(m)$ whenever p is prime, $(m,p)=1$ and $\nu\geqslant 1$. Also by (3.1), we have $D\geqslant 10g$. First, the part of the sum on the left side of (4.4) corresponding to those ℓ with $h(\ell;X)>\sqrt{X}$ has the desired upper bound. Now consider the case $h(\ell;X)\leqslant\sqrt{X}$. Let H be the unique real number satisfying

$$H^{1/2}<h(\ell;X)\leqslant H,\qquad H=(4g)^{2^k}\ \text{ for some non-negative integer } k.$$

Fix H and consider the numbers ℓ corresponding to H. Decompose each ℓ uniquely as

$$\ell=\ell_1\ell_2,\ P^+(\ell_1)\leqslant H<P^-(\ell_2).$$

By the definition (4.1) of $h()$, $\ell_1>\sqrt{X}$. We also have

$$f(\ell)\leqslant f(\ell_1)(4g)^{\Omega(\ell_2)}.$$

Taking $\kappa=4g+4$, and encode the condition $\ell_1>\sqrt{X}$ by introducing a factor $\left(\frac{\log\ell_1}{\log X^{1/2}}\right)^\kappa$. Since $H\geqslant 4g$,

$$\begin{aligned}\sum_{\substack{\ell>\sqrt{X}\\ P^+(\ell)\leqslant z\\ H^{1/2}<h(\ell;X)\leqslant H}} \frac{f(\ell)}{\ell\log^2 h(\ell;X)} &\ll \frac{1}{\log^2 H}\sum_{\substack{\ell_1>\sqrt{X}\\ P^+(\ell_1)\leqslant H}}\frac{f(\ell_1)}{\ell_1}\sum_{\substack{P^+(\ell_2)\leqslant z\\ P^-(\ell_2)>H}}\frac{(4g)^{\Omega(\ell_2)}}{\ell_2}\\ &\ll \frac{1}{\log^2 H}\sum_{P^+(\ell_1)\leqslant H}\frac{f(\ell_1)}{\ell_1}\Big(\frac{\log\ell_1}{\log X^{1/2}}\Big)^\kappa\Big(\frac{\log z}{\log H}\Big)^{4g}\\ &=\frac{(\log z)^{4g}}{(\log H)^{4g+2}(\log X^{1/2})^\kappa}\sum_{P^+(a)\leqslant H}\frac{f(a)\log^\kappa a}{a}.\end{aligned}$$

For the final sum on the right side, the argument in Lemma 3.3 in [**9**] or Lemma 2.2 in [**17**] gives

$$\sum_{\substack{P^+(a)\leqslant H\\ P^-(a)>D}}\frac{f(a)\log^\kappa a}{a} \ll (4g)^\kappa(\log H)^\kappa\sum_{\substack{P^+(b)\leqslant H\\ P^-(b)>D}}\frac{f(b)}{b}.$$

Finally, sum over H and recall that $X \leqslant z$. We obtain

$$\sum_{\substack{\ell > X^{1/2} \\ P^+(\ell) \leqslant z \\ h(\ell;X) \leqslant X^{1/2}}} \frac{f(\ell)}{\ell \log^2 h(\ell;X)} = \sum_{k : X^{1/2^k} \geqslant 2} \sum_{\substack{X^{1/2} < \ell \leqslant zX \\ P^+(\ell) \leqslant z \\ X^{1/2^{k+1}} < h(\ell;X) \leqslant X^{1/2^k}}} \frac{f(\ell)}{\ell \log^2 h(\ell;X)}$$

$$\ll \sum_{k=1}^{\infty} \frac{(\log z)^{4g}}{(\log X^{1/2^k})^{4g+2} (\log X^{1/2})^{\kappa}} (\log X^{1/2^k})^{\kappa} \sum_{P^+(b) \leqslant X^{1/2^k}} \frac{f(b)}{b}$$

$$\ll \frac{1}{\log^2 X} \left(\frac{\log z}{\log X} \right)^{4g} \Big(\sum_{k=1}^{\infty} 2^{-k(\kappa - 4g - 2)} \sum_{P^+(b) \leqslant X} \frac{f(b)}{b} \Big)$$

$$\ll \frac{1}{\log^2 X} \left(\frac{\log z}{\log X} \right)^{4g} \sum_{P^+(b) \leqslant z} \frac{f(b)}{b}.$$

This completes the proof of (4.4).

Next, we remove the squarefull part of a from the sum. Each $a \in \mathbb{N}$ may be uniquely decomposed as $a = a_1 a_2$, where $(a_1, a_2) = 1$, a_1 is squarefree and a_2 is squarefull. As ρ and ϕ_F are multiplicative, $L(a;\eta) \leqslant \tau(a_2) L(a_1;\eta)$ by Lemma 2.6 (i). Recalling (2.1) and (3.2), we see that

$$\sum_{\substack{P^-(a) > D \\ P^+(a) \leqslant z}} \frac{L(a;\eta)\rho(a)}{\varphi_F(a)} \leqslant \sum_{a_1 \in \mathscr{P}(D,z)} \frac{L(a_1;\eta)\rho(a_1)}{\varphi_F(a_1)} \sum_{P^-(a_2) > D} \frac{\tau(a_2)\rho(a_2)}{\varphi_F(a_2)}$$

$$\leqslant \sum_{a_1 \in \mathscr{P}(D,z)} \frac{L(a_1;\eta)\rho(a_1)}{\varphi_F(a_1)} \prod_{p > D} \left(1 + \frac{6g}{p^2} + \frac{8g}{p^3} + \cdots \right)$$

$$\ll \sum_{a_1 \in \mathscr{P}(D,z)} \frac{L(a_1;\eta)\rho(a_1)}{\varphi_F(a_1)}. \tag{4.5}$$

This proves (4.2).

Next, break the sum on the left side of (4.3) into two parts, corresponding to $a \leqslant z^{1/2}$ and to $a > z^{1/2}$. In the first part, $\log^2(P^+(a) + z^{3/4}/a) \gg \log^2 z$, and the desired bound follows from (4.5). Since $H(\ell;X) \leqslant P^+(\ell)$, the second part is majorized by the left side of (4.2) with $X = z$, and thus we see that (4.3) follows from (4.2). □

4.2. **Proof of Proposition 4.1.** Let $\mathcal{A}$ be the set of positive integers $n \leqslant x$ satisfying

(i) $\tau(F(n); y, z) \geqslant 1$;

(ii) $n > \frac{x}{(\log y)^{C+2}}$;

(iii) if p is prime with $p | F(n)$ and $(\log y)^{C+2} < p \leqslant z$, then $p^2 \nmid F(n)$;

(iv) $\displaystyle\prod_{\substack{p^\nu \| F(n) \\ p \leqslant (\log y)^{C+2}}} p^\nu \leqslant \exp\{(\log\log z)^3\}$.

The number of integers $n \leqslant x$ not satisfying (iii) is at most

$$\sum_{(\log y)^{C+2}<p\leqslant z} \Big(\frac{x\rho(p^2)}{p^2} + \rho(p^2)\Big) \ll x \sum_{p>(\log y)^{C+2}} \frac{1}{p^2} + z \ll \frac{x}{(\log y)^{C+2}}.$$

By Lemma 2.4, the number of integers $n \leqslant x$ failing (iv) is

$$\#\{n \leqslant x : \prod_{\substack{p^\nu || F(n) \\ p\leqslant (\log y)^{C+2}}} p^\nu > \exp\{(\log\log z)^3\}\} \ll x \exp\Big\{ -c_4 \frac{(\log\log z)^3}{\log((\log y)^{C+2})}\Big\} \ll \frac{x}{(\log y)^{C+2}}.$$

So we have

$$H_F(x,y,z) \leqslant |\mathcal{A}| + O\bigg(\frac{x}{(\log y)^{C+2}}\bigg). \tag{4.6}$$

Each integer $d \in (y,z]$ has a unique decomposition

$$d = d_0 d_1, \quad P^+(d_0) \leqslant D < P^-(d_1). \tag{4.7}$$

If $d \in (y,z]$ and $d|F(n)$, then by (iv), we have $P^+(d) > (\log y)^{C+2}$ since $z^{1/2} \leqslant y < d$. It follows that $d_1 > 1$. Also, by (iv),

$$d_0 \leqslant y^{1/10}. \tag{4.8}$$

Let

$$X := \min\{z, x^{\delta/2}\}. \tag{4.9}$$

For each $d \in (y,z]$ with $\rho(d) > 0$, let

$$\mathcal{A}_d := \{n \in \mathcal{A} : d|F(n)\}.$$

For each d and $n \in \mathcal{A}_d$, by (iii) and (iv) $F(n)$ has a unique decomposition in the form

$$F(n) = Q_{n,d} M_{n,d} A_{n,d} B_{n,d}, \tag{4.10}$$

with the conditions

$$Q_{n,d}|Q^\infty,\ M_{n,d}|d_1^\infty,\ (A_{n,d}B_{n,d}, Qd_1) = 1 \text{ and } P^+(A_{n,d}) < P^-(B_{n,d}), \tag{4.11}$$

where we choose $A_{n,d}$ as large as possible such that

$$A_{n,d} \leqslant X \text{ and } P^+(A_{n,d}) < P^+(d). \tag{4.12}$$

In particular, $d_0|Q_{n,d}$ and $d_1|M_{n,d}$.

Now fix d and suppose that $n \in \mathcal{A}_d$. Define

$$p = P^+(d).$$

Then by (iv),

$$p = P^+(d_1) \text{ and } p > (\log y)^{C+2}.$$

Write

$$\ell = (d_1/p)A_{n,d}, \tag{4.13}$$

where d_1 and $A_{n,d}$ are defined as in (4.7) and (4.10) under the constraints (4.11) and (4.12). Thus we derive from (iii), (iv) and (4.12) that $P^+(\ell) < p$. Moreover, it is easy to see from (4.7) and (4.13) that $\tau(pd_0\ell; y, z) \geqslant 1$, which implies

$$\log(y/p) \in \mathscr{L}(d_0\ell; \eta).$$

Next, set

$$\vartheta = \frac{\log X}{2\log z}, \tag{4.14}$$

so that by (4.9) we have

$$1 \ll \vartheta \leqslant \frac{1}{2}. \tag{4.15}$$

Partition the set $\mathcal{A}_d$ into the disjoint sets

$$\mathcal{A}_{d,1} := \{n \in \mathcal{A}_d : P^-(B_{n,d}) > p^\vartheta\}, \qquad \mathcal{A}_{d,2} := \{n \in \mathcal{A}_d : P^-(B_{n,d}) \leqslant p^\vartheta\}.$$

First, we consider the set $\bigcup_{d\in(y,z]} \mathcal{A}_{d,1}$. By (4.9), we have

$$p^\vartheta \leqslant z^{1/2} \leqslant x^{1/2}, \qquad pd_0\ell = d_0 d_1 A_{n,d} \leqslant zX < x^{1-\delta/2}$$

and

$$P^+(pd_0\ell) = p = (p^\vartheta)^{1/\vartheta}.$$

Given p, ℓ and d_0 with $p\ell d_0 | F(n)$, it follows that all prime factors of $F(n)$ either divide $p\ell Q$ or are greater than p^ϑ. By Lemma 2.3 with $K' = 1/\vartheta$ and $K = \delta/2$, together with (4.15), we obtain

$$\begin{aligned}\Big|\bigcup_{d\in(y,z]} \mathcal{A}_{d,1}\Big| &\leqslant \sum_{\substack{d_0\ell\leqslant zX\\ P^+(\ell)\leqslant z}} \sum_{\substack{P^+(\ell)<p\leqslant z\\ p\geqslant y/\ell d_0\\ \log(y/p)\in\mathscr{L}(d_0\ell;\eta)}} \sum_{\substack{n\leqslant x, pd_0\ell|F(n)\\ q|F(n)\Rightarrow q|p\ell Q \text{ or } q>p^\vartheta}} 1\\ &\ll \sum_{\substack{d_0\ell\leqslant zX\\ P^+(\ell)\leqslant z}} \sum_{\substack{P^+(\ell)<p\leqslant z\\ p\geqslant y/\ell d_0\\ \log(y/p)\in\mathscr{L}(d_0\ell;\eta)}} \frac{x}{\log p}\frac{\rho(d_0)}{d_0}\frac{\rho(\ell)}{\varphi_F(\ell)}\frac{\rho(p)}{p}.\end{aligned}$$

Since $\mathscr{L}(a;\eta)$ is the disjoint union of intervals of length between $\eta/2$ and η, repeated use of Lemma 2.1 gives

$$\sum_{\substack{P^+(\ell)<p\leqslant z\\ p\geqslant y/\ell d_0\\ \log(y/p)\in\mathscr{L}(d_0\ell;\eta)}} \frac{\rho(p)}{p\log p} \ll \frac{L(d_0\ell;\eta)}{\log^2(y/d_0\ell + P^+(\ell))}.$$

Using Lemma 2.6 (i), $L(d_0\ell;\eta) \leqslant L(\ell;\eta)\tau(d_0)$. Hence, by (4.8),

$$\Big|\bigcup_{d\in(y,z]} \mathcal{A}_{d,1}\Big| \ll x \sum_{P^+(\ell)\leqslant z} \frac{L(\ell;\eta)\rho(\ell)}{\varphi_F(\ell)\log^2(y^{9/10}/\ell + P^+(\ell))} \sum_{d_0|Q^\infty} \frac{\tau(d_0)\rho(d_0)}{\varphi_F(d_0)}.$$

By (2.1),

$$\sum_{d_0|Q^\infty} \frac{\rho(d_0)\tau(d_0)}{d_0} = \prod_{p\leqslant D} \Big(\sum_{\nu=0}^{\infty} \frac{\tau(p^\nu)\rho(p^\nu)}{p^\nu} \Big) \ll 1. \tag{4.16}$$

Using (4.3) and recalling (4.9), we have

$$\Big| \bigcup_{d\in(y,z]} \mathcal{A}_{d,1} \Big| \ll x \sum_{\substack{d_0\ell\leqslant zX \\ P^+(\ell)\leqslant z}} \frac{L(\ell;\eta)\tau(d_0)\rho(\ell)}{\varphi_F(\ell)\log^2(y/d_0\ell + P^+(\ell))} \ll \frac{x}{\log^2 y} \sum_{a\in\mathscr{P}(D,z)} \frac{L(a;\eta)\rho(a)}{\varphi_F(a)}. \tag{4.17}$$

Next, we estimate the size of $\bigcup_{d\in(y,z]} \mathcal{A}_{d,2}$. If $P^-(B_{n,d}) \leqslant p^\vartheta$, then by the definition of $A_{n,d}$, we obtain $A_{n,d}P^-(B_{n,d}) > X$. Hence by ((4.13)) and (4.14), we have

$$\ell \geqslant A_{n,d} > X/P^-(B_{n,d}) \geqslant Xp^{-\vartheta} \geqslant Xz^{-\vartheta} \geqslant X^{1/2}. \tag{4.18}$$

Recalling the definition 4.1 of $h(\cdot)$, we see that $h(\ell;X)$ and $h(A_{n,d};X)$ are well-defined. Then by (4.13) and (4.18), we have

$$h(\ell;X) \leqslant h(A_{n,d};X) \leqslant P^+(A_{n,d}) < P^-(B_{n,d}) \leqslant p^\vartheta < p,$$

where we invoked (4.15) in the last step. Hence, Lemma 2.3 implies (in the sums, $d_0|Q^\infty$ and $(\ell,Q)=1$)

$$\begin{aligned}\Big| \bigcup_{d\in(y,z]} \mathcal{A}_{d,2} \Big| &\leqslant \sum_{\substack{d_0\ell\leqslant zX \\ \ell>X^{1/2} \\ P^+(\ell)\leqslant z}} \sum_{\substack{P^+(\ell)<p\leqslant z \\ p\geqslant y/d_0\ell \\ \log(y/p)\in\mathscr{L}(d_0\ell)}} \sum_{\substack{n\leqslant x, pd_0\ell|F(n) \\ q|F(n)\Rightarrow q|p\ell Q \text{ or } q>h(\ell;X)}} 1 \\ &\ll \sum_{\substack{d_0\ell\leqslant zX \\ \ell>X^{1/2} \\ P^+(\ell)\leqslant z}} \sum_{\substack{P^+(\ell)<p\leqslant z \\ p\geqslant y/d_0\ell \\ \log(y/p)\in\mathscr{L}(d_0\ell)}} \frac{x}{\log h(\ell;X)} \frac{\rho(d_0)}{d_0} \frac{\rho(\ell)}{\varphi_F(\ell)} \frac{\rho(p)}{p}.\end{aligned}$$

As above, applying Lemma 2.1 repeatedly, we obtain

$$\sum_{\substack{P^+(\ell)<p\leqslant z \\ p\geqslant y/d_0\ell \\ \log(y/p)\in\mathscr{L}(d_0\ell)}} \frac{\rho(p)}{p} \ll \frac{L(d_0\ell;\eta)}{\log(P^+(\ell) + \frac{y}{d_0\ell})} \leqslant \frac{L(d_0\ell;\eta)}{\log h(\ell;X)},$$

since $h(\ell;X) \leqslant P^+(\ell)$ (cf., the definition (4.1) of $h()$). Thus,

$$\Big| \bigcup_{d\in(y,z]} \mathcal{A}_{d,2} \Big| \ll x \sum_{\substack{d_0\ell\leqslant zX \\ \ell>X^{1/2} \\ P^+(\ell)\leqslant z}} \frac{\rho(d_0)\rho(\ell)L(d_0\ell;\eta)}{d_0\varphi_F(\ell)\log^2 h(\ell;X)} \leqslant x \sum_{\substack{\ell>X^{1/2} \\ P^+(\ell)\leqslant z}} \frac{\rho(\ell)L(\ell;\eta)}{\varphi_F(\ell)\log^2 h(\ell;X)} \sum_{d_0|Q^\infty} \frac{\rho(d_0)\tau(d_0)}{d_0}, \tag{4.19}$$

where we used Lemma 2.6 (i) in the last step. Thus, applying (4.16) and (4.2) to the right side of (4.19), we find that

$$\left|\bigcup_{d\in(y,z]}\mathcal{A}_{d,2}\right| \ll \frac{x}{\log^2 z}\sum_{a\in\mathscr{P}(D,z)}\frac{L(a;\eta)\rho(a)}{\varphi_F(a)}. \tag{4.20}$$

Finally, we combine (4.6), (4.17) and (4.20) to obtain

$$H_F(x,y,z) \ll \frac{x}{\log^2 y}\sum_{a\in\mathscr{P}(D,z)}\frac{L(a;\eta)\rho(a)}{\varphi_F(a)} + O\left(\frac{x}{(\log y)^{C+2}}\right).$$

The error term is negligible as

$$\sum_{a\in\mathscr{P}(D,z)}\frac{L(a;\eta)\rho(a)}{\varphi_F(a)} \geqslant L(1;\eta) = \eta \gg \frac{1}{(\log y)^C}.$$

This completes the proof of Proposition 4.1. □

5. The upper bound in Theorem 1, Part II

In this section, we complete the proof of the upper bound in Theorem 1 using Proposition 4.1. This part of the argument follows [**7**] with only trivial modification. Recall the partition of the primes larger than D from Section 3, in particular (3.9) and (3.10). The following is analogous to Lemma 3.5 in [**7**].

Lemma 5.1. *Suppose $y \to \infty$, $z - y \to \infty$ and $0 < \eta \leqslant \log y$. Let*

$$v = \left\lceil \frac{\log\log z}{\log 2}\right\rceil$$

and suppose $0 \leqslant k \leqslant 10v$. Then

$$T_k(z) := \sum_{\substack{a\in\mathscr{P}(D,z)\\ \omega(a)=k}}\frac{L(a;\eta)\rho(a)}{\varphi_F(a)} \ll (\eta+1)(2v\log 2)^k U_k(v;\min(1,\eta)),$$

where

$$U_k(v;t) = \int_{0\leqslant\xi_1\leqslant\cdots\leqslant\xi_k\leqslant 1}\min_{0\leqslant j\leqslant k} 2^{-j}\big(2^{v\xi_1}+\cdots+2^{v\xi_j}+t\big)d\boldsymbol{\xi}.$$

Proof. Consider $a = p_1\cdots p_k$ with $D < p_1 < \cdots < p_k \leqslant z$ and define j_i by $p_i \in E_{j_i}$ $(1 \leqslant i \leqslant k)$. Put $l_i = \frac{\log\log p_i}{\log 2}$. By Lemma 2.6 (ii) and (3.10),

$$L(a;\eta) \leqslant 2^k \min_{0\leqslant i\leqslant k} 2^{-i}(2^{l_1}+\cdots+2^{l_i}+\eta) \leqslant (\eta+1)2^{k+c_5}F(\mathbf{j}),$$

where

$$F(\mathbf{j}) = \min_{0\leqslant i\leqslant k} 2^{-i}(2^{j_1}+\cdots+2^{j_i}+\min(1,\eta)).$$

Let J denote the set of vectors $\mathbf{j}$ satisfying $0 \leqslant j_1 \leqslant \cdots \leqslant j_k \leqslant v + c_5 - 1$. Then

$$T_k(z) \leqslant (\eta+1)2^{k+c_5} \sum_{\mathbf{j}\in J} F(\mathbf{j}) \sum_{\substack{D<p_1<\cdots<p_k \\ p_i\in E_{j_i}\ (1\leqslant i\leqslant k)}} \frac{\rho(p_1\cdots p_k)}{\varphi_F(p_1\cdots p_k)}. \tag{5.1}$$

If b_j is the number of primes p_i in E_j for $1 \leqslant j \leqslant v + c_5 - 1$, the sum over $p_1, ..., p_k$ above is at most

$$\prod_{j=1}^{v+c_5-1} \frac{1}{b_j!}\Big(\sum_{p\in E_j} \frac{\rho(p)}{\varphi_F(p)}\Big)^{b_j} \leqslant ((v+c_5)\log 2)^k \int_{R(\mathbf{j})} 1 d\xi \leqslant e^{10c_5}(v\log 2)^k \int_{R(\mathbf{j})} 1 d\xi,$$

where

$$R(\mathbf{j}) = \{0 \leqslant \xi_1 \leqslant \cdots \leqslant \xi_k \leqslant 1 : j_i \leqslant (v+c_5)\xi_i \leqslant j_i + 1\ \forall i\} \subseteq R_k.$$

Finally, since $2^{j_i} \leqslant 2^{(v+c_5)\xi_i} \leqslant 2^{c_5}2^{v\xi_i}$ for each i,

$$\sum_{\mathbf{j}\in J} F(j) \int_{R(\mathbf{j})} 1 d\xi \leqslant 2^{c_5} U_k(v; \min(1,\eta)).$$

So by (5.1), we obtain

$$T_k(z) \ll (\eta+1)(2v\log 2)^k U_k(v; \min(1,\eta)). \qquad \square$$

When $z_0(y) \leqslant z \leqslant y^{1+\delta/2}$, where $z_0(y)$ is defined in (2.5), the upper bound in Lemma 5.1 is identical to the bound in [**7**, Lemma 3.5] (taking $Q = 1$ in this lemma). Therefore, the proof of Lemma 3.7 in [**7**] provides the required upper bound for $\sum_k T_k(z)$. Combined with Proposition 4.1 (replacing δ with $\delta/2$), this gives the desired upper bound for $H_F(x,y,z)$ in Theorem 1. When $y + y/(\log y)^C \leqslant z \leqslant z_0(y)$, the upper bound follows form the simple estimate

$$H_F(x,y,z) \ll \sum_{y<d\leqslant z} \frac{\rho(d)x}{d} \ll \eta x,$$

a consequence of Lemma 2.2. Finally, when $z \geqslant y^{1+\delta/2}$, the trivial bound $H_F(x,y,z) \leqslant x$ suffices.

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