Divisors of the Euler and Carmichael functions

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1 Introduction

Two of the most studied functions in the theory of numbers are Euler's totient function $\phi(n)$ and Carmichael's function $\lambda(n)$, the first giving the order of the group $(\mathbb{Z}/n\mathbb{Z})^*$ of reduced residues modulo n, and the latter giving the maximum order of any element of $(\mathbb{Z}/n\mathbb{Z})^*$. The distribution of $\phi(n)$ and $\lambda(n)$ has been investigated from a variety of perspectives. In particular, many interesting properties of these functions require knowledge of the distribution of prime factors of $\phi(n)$ and $\lambda(n)$, e.g., [3], [5], [4], [6], [7], [12], [19].

The distribution of all of the divisors of $\phi(n)$ and $\lambda(n)$ has thus far received little attention, perhaps due to the complicated way in which prime factors interact to form divisors. From results about the normal number of prime factors of $\phi(n)$ and $\lambda(n)$ [5], one deduces immediately that $\tau(\phi(n))$ and $\tau(\lambda(n))$ are each $\exp\{\frac{\log 2}{2}(\log\log n)^2\}$ for almost all n. However, the determination of the *average* size of $\tau(\phi(n))$ and of $\tau(\lambda(n))$ is more complex, and has been studied recently by Luca and Pomerance [13].

In this note we investigate problems about localization of divisors of $\phi(n)$ and $\lambda(n)$. Our results have application to the structure of $(\mathbb{Z}/n\mathbb{Z})^*$, since the set of divisors of $\lambda(n)$ is precisely the set of orders of elements of $(\mathbb{Z}/n\mathbb{Z})^*$. We say that a positive integer m is u-*dense* if whenever $1 \le y < m$, there is a divisor of m in the interval $(y, uy]$. The distribution of u-dense numbers for general u has been investigated by Tenenbaum ([17], [18]) and Saias ([14], [15]). According to Théorème 1 of [14], the number of u-dense integers $m \le x$ is $\asymp (x \log u)/\log x$, uniformly for $2 \le u \le x$. In particular, the number of 2-dense integers $m \le x$ is $\asymp x/\log x$, that is, the 2-dense integers are about as sparse as the primes.

By contrast, we show that 2-dense values of $\phi(n)$ and $\lambda(n)$ are very common.

Theorem 1. *If x is sufficiently large, then for $\gg x$ integers $n \le x$, both $\phi(n)$ and $\lambda(n)$ are 2-dense.*

There are relatively simple heuristic reasons for believing Theorem 1. Recall that

$$\phi(p_1^{e_1}\cdots p_k^{e_k}) = p_1^{e_1-1}(p_1-1)\cdots p_k^{e_k-1}(p_k-1),$$
$$\lambda(p_1^{e_1}\cdots p_k^{e_k}) = \text{lcm}[\lambda(p_1^{e_1}),\dots,\lambda(p_k^{e_k})],$$

where $\lambda(p_i^{e_i}) = \phi(p_i^{e_i})$ if p_i is odd or $p_i = 2$ and $e_i \le 2$, and $\lambda(2^e) = 2^{e-2}$ for $e \ge 3$. In particular, $\phi(n)$ and $\lambda(n)$ have the same prime factors. Most of these prime factors are factors of shifted primes $p-1$ where $p|n$, and thus it is important to understand the distribution of prime factors of shifted primes. By classical results in probabilistic number theory (see, e.g., Theorem 10 of [10]), most numbers $n \le x$ have about $\log\log x$ prime factors, roughly uniformly distributed on a $\log\log -$scale. For most primes p, $p-1$ has about $\log\log p$ prime factors [3], that is, the multiplicative structure of a typical shifted prime $p-1 \le x$ is similar to the multiplicative structure of a typical integer $m \le x$. Thus, we find that for most values of n,

$$\Omega(\phi(n)) \approx \sum_{k\le\log\log n} k \approx \frac{1}{2}(\log\log n)^2,$$

where $\Omega(m)$ is the number of prime power divisors of m (see [5] for a precise result of this kind). We have $\Omega(m) \approx \log\log x$ for most $m \le x$, so usually $\phi(n)$ has far more divisors than a typical integer of its size. We therefore expect the divisors of $\phi(n)$, especially the smaller divisors, to be "very dense" for most n, and the same should be true of small divisors of $\lambda(n)$. On the other hand, there are a large proportion of n for which the divisors of $\phi(n)$ and $\lambda(n)$ are not very dense. To state our next result, we define θ to be the supremum of real numbers c so that there are $\gg x/\log x$ primes $p \le x$ with $p-1$ having a prime factor $> p^c$. Many papers have been written on bounding θ, and the current record is $\theta \ge 0.677$ and due to Baker and Harman [1].

Theorem 2. *Let $0 < c < 2\theta - 1$. If x is sufficiently large, then for $\gg_c x$ of the integers $n \le x$, neither $\phi(n)$ nor $\lambda(n)$ is x^c-dense.*

It is conjectured that $\theta = 1$, and this would imply the conclusion of Theorem 2 for any $c < 1$.

If $u < 2$, there are no u-dense integers $m > 1$. However, it is possible that the divisors of a given integer in some long interval do have consecutive ratios which are $\le u$. We say that an integer n is u-dense in a set I if for every $y \in I$, the interval $(y, uy]$ contains a divisor of n. The following makes precise what we claimed earlier about the "very dense" nature of the small divisors of $\phi(n)$ and $\lambda(n)$.

Theorem 3. *For every positive integer h and $0 < \delta < 1$, there is a constant $c = c(h,\delta) > 0$ so that if x is sufficiently large, then for more than $(1-\delta)x$ of the integers $n \le x$, $\phi(n)$ and $\lambda(n)$ are both $(1+1/h)$-dense in $[h, x^c]$.*

Notice that the left endpoint h of the interval cannot be replaced by $h-1$, since if $h-1 \le a < h/(1+1/h)$, there are no integers in $(a, a(1+1/h)]$. Likewise, if we assume that $\theta = 1$, then we cannot take c independent of δ in light of Theorem 2.

Using Theorem 3, we prove a more general version of Theorem 1.

Theorem 4. *For every positive integer h, there are $\gg_h x$ integers $n \le x$ such that $\phi(n)$ is $(1+1/h)$-dense in $[h, \phi(n)/(h+1))$ and $\lambda(n)$ is $(1+1/h)$-dense in $[h, \lambda(n)/(h+1))$.*

We also record a limiting case of Theorem 3.

Corollary 1. *Suppose $g(x)$ is a positive function decreasing monotonically to 0 and let h be a positive integer. Almost all $n \le x$ have the property that $\phi(n)$ and $\lambda(n)$ are $(1+1/h)$-dense in $[h, x^{g(x)}]$.*

Analogous to the problems studied in [9], [8], [16], we can study the distribution of integers with $\phi(n)$ having a divisor in a *single* interval. Let

$$B(x, y, z) = |\{n \le x : \exists\, d | \phi(n), y < d \le z\}|.$$

An almost immediate corollary of Theorems 1, 2 and 3 is the following result in the special case $z = 2y$.

Corollary 2.

(i) Uniformly for $1 \le y \le x/2$, we have $B(x, y, 2y) \gg x$.

(ii) Fix $1-\theta < c < 1/2$. Then, uniformly for $x^c \le y \le x^{1-c}$, we have $x - B(x, y, 2y) \gg x$.

(iii) Let $g(x) \to 0$ monotonically. Then, for $1 \le y \le x^{g(x)}$, we have $B(x, y, 2y) \sim x$.

We leave as an open problem the determination of the order of magnitude of $B(x, y, z)$ for all x, y, z.

We note that easy modifications of our proofs give the same results for the sum of divisors function $\sigma(n)$ in place of $\phi(n)$, since $\sigma(p) = p + 1$ for primes p.

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2 Preliminaries

Throughout this paper, the letters p and q, with or without subscripts, will always denote primes. Constants implied by the O and $\ll$ symbols are absolute, unless dependence on a parameter is indicated by a subscript. All constants are effectively computable as well. We denote by $P^+(m)$ the largest prime factor of m, with the convention that $P^+(1) = 0$.

Our key lemma, presented below, says roughly that the small *prime* factors of $\phi(n)$ are quite dense.

Lemma 2.1. *For some large constant C, if $C/\log x \le g \le 1/10$ and $1/(g\log x) \le \varepsilon \le \frac{1}{4}$, then the number of $n \le x$ for which $\phi(n)$ does not have a prime divisor in $(x^g, x^{g(1+\varepsilon)}]$ is $\ll g^{\varepsilon/2}\log(1/g)x$.*

Proof. First, note that the conclusion is trivial if $\varepsilon\log(1/g) \le 1$, hence we may assume that $\varepsilon\log(1/g) \ge 1$. Next we claim that for large x and $w \ge x^{6g}$, that

$$|\{p \le w : p-1 \text{ has no prime factor in } (x^g, x^{g(1+\varepsilon)}]\}| \le \left(1 - \frac{2\varepsilon}{3}\right)\frac{w}{\log w}. \tag{2.1}$$

Let $\pi(w; q, a)$ be the number of primes $p \le w$ which satisfy $p \equiv a \pmod{q}$. For positive integer q, write

$$\pi(w; q, 1) = \frac{\mathrm{li}(w)}{\phi(q)} + E(w; q),$$

where

$$\mathrm{li}(w) = \int_2^w \frac{dt}{\log t}.$$

Using the Bombieri-Vinogradov Theorem ([2], Ch. 28) and the Mertens' estimates, the number of primes $p \le w$ that $p-1$ does have a prime factor in $(x^g, x^{g(1+\varepsilon)}]$ is

$$\begin{aligned}
&\ge \sum_{x^g < q \le x^{(1+\varepsilon)g}} \pi(w; q, 1) - \sum_{x^g < q_1 < q_2 \le x^{(1+\varepsilon)g}} \pi(w; q_1q_2, 1) \\
&= \sum_q \left(\frac{\mathrm{li}(w)}{q-1} + E(w; q)\right) - \sum_{q_1,q_2}\left(\frac{\mathrm{li}(w)}{(q_1-1)(q_2-1)} + E(w; q_1q_2)\right) \\
&= \mathrm{li}(w)\left[\log(1+\varepsilon) - \frac{1}{2}\log^2(1+\varepsilon) + O\left(\frac{1}{\log^2 x^g}\right)\right] + O\left(\frac{w}{\log^3 w}\right) \\
&\ge \frac{3\varepsilon}{4}\frac{w}{\log w}.
\end{aligned}$$

For the last step, we used the fact that $w \ge x^{6g} \ge e^{6C}$ and C is sufficiently large. This proves (2.1).

Consider $x/\log x < n \le x$ such that $\phi(n)$ does not have a prime divisor in $(x^g, x^{g(1+\varepsilon)}]$. We can write $n = q_1^{\alpha_1} q_2^{\alpha_2} ... q_k^{\alpha_k} m$, where $q_1 > q_2 > \cdots > q_k > x^{6g}$, $\alpha_i \ge 1$ for $1 \le i \le k$ and $P^+(m) \le x^{6g}$. Then $q_1, \dots, q_k \in T$, the set of primes p such that $p-1$ does not have a prime factor in $(x^g, x^{g(1+\varepsilon)}]$. By (2.1) and partial summation,

$$\sum_{\substack{x^{6g} < q \le x \\ q \in T}} \sum_{a \ge 1} \frac{1}{q^a} \le \left(1 - \frac{2\varepsilon}{3}\right)\left(\log \frac{1}{6g} + \frac{1}{\log x}\right) + \sum_{q > e^{6C}} \frac{1}{q(q-1)}$$
$$\le \left(1 - \frac{\varepsilon}{2}\right) \log \frac{1}{6g}$$

for sufficiently large x. By Theorem 07 of [10], for some positive constant c_0 and uniformly in $x \ge z$, $y \ge 2$, the number of integers $n \le x$ divisible by a number $m > z$ with $P^+(m) \le y$ is $\ll x \exp\{-c_0 \frac{\log z}{\log y}\}$. Consequently, the number of n with $m > x^{1/3}$ is $\ll xe^{-c_0/18g} \ll gx$. For other n, we may assume that $m \le x^{1/3}$, and thus $k \ge 1$. Again by the above theorem, the number of n with $q_1 \le \log^{10} x$ is $\ll x/\log x \ll gx$. For remaining n, we have $q_1^{\alpha_1 - 1} \cdots q_k^{\alpha_k - 1} \le \log^2 x$, for otherwise, $q_1^{\lfloor \alpha_1/2 \rfloor} \cdots q_k^{\lfloor \alpha_k/2 \rfloor} \ge q_1^{(\alpha_1-1)/2} \cdots q_k^{(\alpha_k-1)/2} > \log x$ and the number of n divisible by d^2 for some $d > \log x$ is $O(x/\log x)$. Hence $q_1 \cdots q_k \ge x^{1/2}$. In particular, $q_1 \ge \max(x^{\frac{1}{2k}}, \log^{10} x)$ and $\alpha_1 = 1$. Given $q_2^{\alpha_2}, \dots, q_k^{\alpha_k}$, and m, the number of q_1 is, by the Chebyshev estimates for primes,

$$\ll \frac{x}{q_2^{\alpha_2} \cdots q_k^{\alpha_k} m \log(x/(q_2^{\alpha_2} \cdots q_k^{\alpha_k} m))} \ll \frac{kx}{\log x} \frac{1}{q_2^{\alpha_2} \cdots q_k^{\alpha_k} m}.$$

Given $q_2^{\alpha_2}, \dots, q_k^{\alpha_k}$,

$$\sum_{P^+(m) \le x^{6g}} \frac{1}{m} \ll \log(x^{6g}) = 6g \log x.$$

With fixed k, we have

$$\sum_{q_2, \dots, q_k \in T} \sum_{\alpha_2, \dots, \alpha_k \ge 1} \frac{1}{q_2^{\alpha_2} \cdots q_k^{\alpha_k}} \le \frac{1}{(k-1)!} \Bigg(\sum_{\substack{x^{6g} < q \le x \\ q \in T}} \sum_{a \ge 1} \frac{1}{q^a} \Bigg)^{k-1} \le \frac{((1 - \frac{\varepsilon}{2}) \log \frac{1}{6g})^{k-1}}{(k-1)!}.$$

The total number of such n is

$$\begin{aligned}
&\ll gx + gx \sum_{1\le k\le 1/(6g)} \frac{k}{(k-1)!}\left(\left(1-\frac{\varepsilon}{2}\right)\log\frac{1}{6g}\right)^{k-1}\\
&\ll gx + gx\left(\log\frac{1}{6g}\right)\sum_{j=0}^{\infty}\frac{((1-\frac{\varepsilon}{2})\log\frac{1}{6g})^j}{j!}\\
&= gx + gx\left(\log\frac{1}{6g}\right)\left(\frac{1}{6g}\right)^{1-\frac{\varepsilon}{2}}\\
&\ll g^{\frac{\varepsilon}{2}}\left(\log\frac{1}{g}\right)x.
\end{aligned}$$

This completes the proof. □

Remarks. Since $\phi(n)$ and $\lambda(n)$ have the same prime factors, Lemma 2.1 holds with ϕ replaced by λ. With a finer analysis, it is possible to remove the factor $\log(1/g)$ appearing in the conclusion of Lemma 2.1. Also, if ε is fixed, then $g^{\varepsilon/2}\log(1/g) \ll_\varepsilon g^{\varepsilon/3}$, an inequality we shall use in the application of Lemma 2.1.

We next give a method of constructing integers which are dense in an interval.

Lemma 2.2. *Suppose that h is a positive integer, $y \ge h$, and D is $(1+1/h)$-dense in $[h, y]$. Suppose also that $m = Dm_1\cdots m_k$, where for $1 \le j \le k$, $m_j \le (y/h)m_1\cdots m_{j-1}$. Then m is $(1+1/h)$-dense in $[h, m_1\cdots m_k y]$.*

Proof. By hypothesis, the lemma holds for $k = 0$. Suppose the lemma is true for $k = l$, m satisfies the hypotheses with $k = l+1$ and put $m' = Dm_1\cdots m_l$. Then m' is $(1+1/h)$-dense in $[h, m_1\cdots m_l y]$. Multiplying the divisors of m' by m_{l+1}, we find that m is also $(1+1/h)$-dense in $[m_{l+1}h, m_1\cdots m_{l+1}y]$. Our assumption about m_{l+1} implies that m is $(1+1/h)$-dense in $[h, m_1\cdots m_{l+1}y]$, as desired. □

Lemma 2.3. *Given any positive integer D, n is divisible by a prime $q \equiv 1 \pmod D$ for almost all n.*

Proof. By a theorem of Landau [11], the number of $n \le x$ which have no prime factor $q \equiv 1 \pmod D$ is asymptotic to $c(D)x(\log x)^{-1/\phi(D)}$ for some constant $c = c(D)$. □

Luca and Pomerance [12] have recently proven a stronger statement, namely that for some constant c_1, for almost all integers n, $\phi(n)$ is divisible by every prime power $\le c_1\frac{\log\log n}{\log\log\log n}$.

3 Proof of the theorems

Proof of Theorem 3. Fix h and δ, and let y be sufficiently large, depending on h, and such that $y > h^5$. Let D be the product of all prime powers $\le y$. Let $\varepsilon = \frac{1}{4}$ and let $Y = (y/h)^{4/5}$. Let C be the constant in Lemma 2.1.

Consider the intervals $I_j = (Y^{(5/4)^{j-1}}, Y^{(5/4)^j}]$ $(1 \le j \le J)$, where $Y \ge e^C$. Fix c so that $0 < c \le 1/20$, let x be sufficiently large, and take J so that $Y^{(5/4)^{J-2}} < x^c \le Y^{(5/4)^{J-1}}$. Then $Y^{(5/4)^J} < (Y^{(5/4)^{J-2}})^2 < x^{2c} \le x^{1/10}$. By Lemma 2.1, if y is large enough, then the number of integers $n \le x$ for which $\phi(n)$ does not have prime factors in I_j is

$$\ll_\varepsilon \left(\frac{\log Y^{(5/4)^{j-1}}}{\log x} \right)^{1/12} x.$$

Summing over j, we find that $\phi(n)$ has a prime factor in every interval I_j for all $n \le x$ except for a set of size

$$\ll \left(\frac{\log Y^{(5/4)^J}}{\log x} \right)^{1/12} x < (2c)^{1/12} x.$$

If c is small enough, for at least $(1 - \delta/2)x$ of the integers $n \le x$, $\phi(n)$ has a prime factor in every interval I_j. Applying Lemma 2.3, for at least $(1-\delta)x$ integers $n \le x$, $\phi(n)$ is divisible by a prime $q \equiv 1 \pmod{D^3}$ and has a prime factor in every interval I_j. For each such n, let $p_1, \ldots, p_J$ be primes dividing $\phi(n)$ and such that $p_j \in I_j$ for $1 \le j \le J$. By hypothesis, $p_3 > (y/h)^{5/4} > y$, hence $p_j \nmid D$ for $j \ge 3$. Since $D^3 | (q-1) | \lambda(n) | \phi(n)$, we have that $\lambda(n)$ and $\phi(n)$ are each divisible by $Dp_1 \cdots p_J$. By definition, D is divisible by every positive integer $\le y$, hence D is $(1+1/h)$-dense in $[h, y]$. Also, $p_1 \le Y^{5/4} = y/h$, and for $j \ge 2$,

$$p_j \le Y^{(5/4)^j} \le Y^{5/4} \prod_{1 \le i \le j-1} Y^{(5/4)^{i-1}} \le (y/h) p_1 \cdots p_{j-1}.$$

By Lemma 2.2, $\phi(n)$ and $\lambda(n)$ are $(1+1/h)$-dense in $[h, p_1 \cdots p_J y]$. Since $p_J > Y^{(5/4)^{J-1}} \ge x^c$, this concludes the proof. □

Proof of Theorems 1 and 4. Applying Theorem 3, there is a positive integer k so that when z is large enough, for more than half of the positive integers $d \le z$, $\phi(d)$ and $\lambda(d)$ are $(1+1/h)$-dense in $[h, z^{1/k}]$. Put $\varepsilon = \frac{1}{5k^2}$, let x be sufficiently large and $x^{\frac{1}{2}} < d \le x^{\frac{1}{2}+\varepsilon}$, where $\phi(d)$ is $(1+1/h)$-dense in $[h, x^{\frac{1}{2k}}]$. Consider distinct primes $p_1, p_2, \ldots, p_k \in I := [x^{\frac{1}{2k}-2\varepsilon}, x^{\frac{1}{2k}-\varepsilon}]$ which do not divide d. Note that

$$x^{1-2k\varepsilon} \le dp_1 p_2 \cdots p_k \le x^{1-(k-1)\varepsilon}. \tag{3.1}$$

Let q be a prime not dividing $dp_1 \cdots p_k$ and satisfying

$$\frac{1}{2}\frac{x}{dp_1 \cdots p_k} < q \le \frac{x}{dp_1 \cdots p_k}, \tag{3.2}$$

so that by (3.1) and the definition of ε,

$$x^{\frac{1}{6k}} \le q \le x^{\frac{2}{5k}}. \tag{3.3}$$

We claim that for all such numbers $n = dp_1 \cdots p_k q$ satisfying the additional hypothesis

$$\lambda(n) \ge x^{1-\varepsilon}, \tag{3.4}$$

$\phi(n)$ is $(1 + 1/h)$-dense in $[h, \phi(n)/(h + 1))$ and $\lambda(n)$ is $(1 + 1/h)$-dense in $[h, \lambda(n)/(h + 1))$. Let $y = x^{\frac{1}{2k}}$. Observe that $\phi(n) = \phi(d)(p_1 - 1) \cdots (p_k - 1)(q - 1)$, $\phi(d)$ is $(1 + 1/h)$-dense in $[h, y]$, $p_i - 1 \le x^{\frac{1}{2k}-\varepsilon} < (y/h)$ $(1 \le i \le k)$ and $q \le (y/h)$. By Lemma 2.2 with $D = \phi(d)$, $m_i = p_i - 1$ $(1 \le i \le k)$ and $m_{k+1} = q - 1$, $\phi(n)$ is $(1 + 1/h)$-dense in $[h, w]$, where $w = y(p_1 - 1) \cdots (p_k - 1)(q - 1)$. By (3.1) and (3.2),

$$w \ge 2^{-k-1} y p_1 \cdots p_k q \ge 2^{-k-2} \frac{x^{1+\frac{1}{2k}}}{d} \ge h\sqrt{x}.$$

But $\phi(n)$ is also $(1 + 1/h)$-dense in $[\phi(n)/w, \phi(n)/(h + 1))$ since $d|m \iff (m/d)|m$, consequently $\phi(n)$ is $(1 + 1/h)$-dense in $[h, \phi(n)/(h + 1))$.

The argument for $\lambda(n)$ is similar, except that now

$$\lambda(n) = \lambda(d)\frac{q-1}{f}\prod_{i=1}^{k}\frac{p_i - 1}{f_i},$$

where f is some divisor of $q - 1$ and f_i is some divisor of $p_i - 1$ $(1 \le i \le k)$. Here we use (3.4), which implies that $f f_1 \cdots f_k \le x^{\varepsilon}$. By Lemma 2.2 with $D = \lambda(d)$, $m_i = (p_i - 1)/f_i$ $(1 \le i \le k)$ and $m_{k+1} = (q - 1)/f$, we see that $\lambda(n)$ is $(1 + 1/h)$-dense in $[h, w]$, where

$$w = y\frac{q-1}{f}\prod_{i=1}^{k}\frac{p_i - 1}{f_i} \ge 2^{-k-2}\frac{x^{1+\frac{1}{2k}-\varepsilon}}{d} \ge h\sqrt{x}.$$

As with $\phi(n)$, we conclude that $\lambda(n)$ is $(1 + 1/h)$-dense in $[h, \lambda(n)/(h + 1))$.

Notice that for the above n, when $h = 1$, $\phi(n)$ is 2-dense in $[1, \phi(n)/2)$. Since $\phi(n)$ is a divisor of itself, we conclude that $\phi(n)$ is 2-dense in $[1, \phi(n))$ and hence 2-dense. This conclusion also holds for $\lambda(n)$ by similar arguments.

Finally, we show that the number of such integers $n \le x$ is $\gg_h x$. First, (3.4) holds for almost all n by Theorem 2 of [6]. By the prime number theorem and (3.3), given $d, p_1, \dots, p_k$, the number of possible primes q is $\gg_k x/(dp_1 \cdots p_k \log x)$. We also have

$$\sum_{p_1,\dots,p_k \in I} \frac{1}{p_1 \cdots p_k} \gg_k 1,$$

and $\sum 1/d \gg \log x$ by partial summation. Hence, there are $\gg_k x$ tuples $(d, p_1, \dots, p_k, q)$ with product $n \in (x/2, x]$ and with $\phi(n)$ and $\lambda(n)$ being $(1+1/h)$-dense respectively. Given such an integer n, n has at most $6k$ prime factors $\ge x^{\frac{1}{6k}}$, hence the number of tuples $(d, p_1, \dots, p_k, q)$ with product n is bounded by a function of k. Thus the proof is complete. □

Proof of Theorem 2. Suppose $0 < c < 2\theta - 1$, and Let $\varepsilon > 0$ be so small that $2\theta - 1 - 6\varepsilon > c$. Consider $n = pm \le x$, where $x^{1-2\varepsilon} < p \le x^{1-\varepsilon}$, and $P^+(p-1) > p^{\theta-\varepsilon}$. By the definition of θ, there are $\gg z/\log z$ such primes $\le z$, if z is large enough. Then $\phi(n)$ and $\lambda(n)$ are each divisible by a prime q with $q > x^{(1-2\varepsilon)(\theta-\varepsilon)} > x^{\theta-3\varepsilon}$, and therefore neither function has divisors in $[x^{1-\theta+3\varepsilon}, x^{\theta-3\varepsilon}]$. The number of such n is, by partial summation,

$$= \sum_{\substack{x^{1-2\varepsilon} < p \le x^{1-\varepsilon} \\ P^+(p-1) > p^{\theta-\varepsilon}}} \left\lfloor \frac{x}{p} \right\rfloor \gg_\varepsilon x,$$

and the proof is complete. □

Proof of Corollary 1. Let $\delta > 0$. By Theorem 3, if x is sufficiently large, then for at least $(1-\delta)x$ integers $n \le x$, both $\phi(n)$ and $\lambda(n)$ are $(1+1/h)$-dense in $[h, x^{g(x)}]$. Since δ is arbitrary, the corollary follows. □

Proof of Corollary 2. (i) The elementary inequality $\sum_{n \le x} n/\phi(n) \ll x$ implies that

$$|\{n \le x : \phi(n) \le \varepsilon n\}| \ll \varepsilon x \qquad (0 < \varepsilon \le 1).$$

Consequently, using Theorem 1, if c is small enough then there are $\gg x$ of the integers $n \le x$ for which $\phi(n)$ is 2-dense and $\phi(n) \ge cx$. This proves (i) for $y \le cx$. For a given constant $f \in [c, 1/2]$, it is an elementary fact that $fx < \phi(n) \le 2fx$ for $\gg_f x$ integers $n \le x$. This completes the proof for the remaining y.

(ii) From the proof of Theorem 2, for a positive proportion of integers n, $\phi(n)$ has no divisors in $[x^c, x^{1-c}]$.

(iii) This follows immediately from Corollary 1. □

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