THE JUMPING CHAMPIONS OF THE FAREY SERIES

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Abstract. Let $\mathfrak{F}_Q$ be the sequence of Farey fractions of order Q. We study the distribution of differences between consecutive Farey fractions, in particular the differences which occur most frequently.

1. Introduction

Let $\mathcal{M} = \{\gamma_1, \ldots, \gamma_M\}$ be a set of real numbers ordered increasingly and let $D(\mathcal{M}) = \{\gamma_{i+1} - \gamma_i \,:\, 1 \le i \le M-1\}$ be the set of gaps between consecutive elements of $\mathcal{M}$. Usually we think of the elements of $D(\mathcal{M})$ as being arranged in ascending order, keeping in the list all the numbers with their multiplicities. A number d is called a *jumping champion of* $\mathcal{M}$ (for short *champion* or JC) if the multiplicity of d is largest among all elements of $D(\mathcal{M})$. According to [9], the term *jumping champion* was introduced by J.H. Conway in 1993.

Finding the JC of a set may be a very difficult problem. Certainly this is the case when $\mathcal{M} = \mathcal{P}_n$, the set of primes less than or equal to n. This problem has been investigated by Nelson [8], Erdős and Straus [1], and Harley [4]. Assuming conjectures for counts of prime r-tuples, Gallagher [2], [3] proved that $D(\mathcal{P}_n)$ approaches a Poisson distribution as $n \to \infty$. Odlyzko, Rubinstein and Wolf [9] give empirical and heuristic evidences that the JC for primes are 1, 2, 4 and the primorials 6, 30, 210, 2310, ... But the technical difficulties encountered when dealing with consecutive primes are formidable enough, that they cannot give a proof of this result even under the assumption of the prime r-tuple conjecture.

Date: September 10, 2002.

2000 Mathematics Subject Classification: 11B57, 11N37.

Our object here is to study the JC problem for the Farey series. Unlike the case of primes, in the case of Farey fractions we may prove, unconditionally, asymptotics for the size of champions, and understand some of their arithmetical properties.

In the next two sections we introduce some terminology and state our main results, which are then proved in the following sections. At the end of the paper we attached samples from three tables showing the JC 's for Farey sequences and related quantities.

2. Notations

Let $\mathfrak{F}_Q$ be the sequence of Farey fractions of order Q. Taking into account the symmetry of $\mathfrak{F}_Q$ with respect to $1/2$, in what follows we will only work with $\mathcal{M}_Q = \mathfrak{F}_Q \cap [0, 1/2]$. Then the number of gaps between consecutive elements of $\mathcal{M}_Q$ is $|D(\mathcal{M}_Q)| = (|\mathfrak{F}_Q| - 1)/2$.

Looking at $1 \le Q \le 9$, one finds that all the elements of $D(\mathcal{M}_Q)$ are distinct, so they all share the position of champion. When $Q = 10$, we have

$$\mathcal{M}_{10} = \left\{0, \frac{1}{10}, \frac{1}{9}, \frac{1}{8}, \frac{1}{7}, \frac{1}{6}, \frac{1}{5}, \frac{2}{9}, \frac{1}{4}, \frac{2}{7}, \frac{3}{10}, \frac{1}{3}, \frac{3}{8}, \frac{2}{5}, \frac{3}{7}, \frac{4}{9}, \frac{1}{2}\right\}$$

and

$$D(\mathcal{M}_{10}) = \left\{\frac{1}{10}, \frac{1}{90}, \frac{1}{72}, \frac{1}{56}, \frac{1}{42}, \frac{1}{30}, \frac{1}{45}, \frac{1}{36}, \frac{1}{28}, \frac{1}{70}, \frac{1}{30}, \frac{1}{24}, \frac{1}{40}, \frac{1}{35}, \frac{1}{63}, \frac{1}{18}\right\}.$$

The gap $1/30$ appears twice in $D(\mathcal{M}_{10})$, so it is the JC of $\mathfrak{F}_{10}$.

We know that if $\frac{a'}{q'}, \frac{a''}{q''}$ are consecutive fractions in $\mathfrak{F}_Q$, then the gap between them is $\frac{a''}{q''} - \frac{a'}{q'} = \frac{1}{q'q''}$, so we need to focus mainly on the pairs of consecutive denominators of Farey fractions. Moreover, there is a bijection (see [5]) between these pairs and the lattice points from

$$\mathcal{T}_Q = \left\{(q_1, q_2) :\ 1 \le q_1, q_2 \le Q,\ q_1 + q_2 > Q,\ \gcd(q_1, q_2) = 1\right\}.$$

This motivates the following definitions. Denote by $h(D, Q)$ the number of gaps of length $1/D$ in $\mathcal{M}_Q$, and this can be written as:

$$\begin{aligned} h(D, Q) &= \left|\left\{(q_1, q_2) \in \mathcal{T}_Q :\ q_1 q_2 = D,\ q_1 < q_2\right\}\right| \\ &= \left|\left\{q | D :\ \gcd(q, \tfrac{D}{q}) = 1,\ \tfrac{D}{q} < q \le Q,\ \tfrac{D}{q} + q > Q\right\}\right|. \end{aligned}$$

It is plain that

$$\left|D(\mathcal{M}_Q)\right| = \sum_{D\geq 1} h(D,Q)\,.$$

Then any JC is a solution of the maximum problem:

$$M(Q) = \max_D h(D,Q)\,.$$

We set

$$\text{Champs}(Q) = \big\{D:\ \ h(D,Q) = M(Q)\big\}.$$

Usually we refer to the elements of Champs(Q) as being champions, although, strictly speaking, the champions are the inverses of the elements of Champs(Q). Next, we denote by $H(Q)$ the number of distinct gaps (participants in the competition for JC), i.e.

$$H(Q) = \left|\{D:\ \ h(D,Q)\geq 1\}\right|.$$

Then we define $H_r(Q)$ to be the number of gaps with multiplicity r, that is

$$H_r(Q) = \left|\{D:\ \ h(D,Q) = r\}\right|.$$

Clearly

$$H(Q) = \sum_{r\geq 1} H_r(Q)\,.$$

We also consider $G_r(Q)$, the number of gaps with multiplicity $\geq r$, or

$$G_r(Q) = \Big|\big\{D:\ \ h(D,Q)\geq r\big\}\Big| = \sum_{i\geq r} H_i(Q)\,.$$

Note that $G_1(Q) = H(Q)$. There is another equivalent definition of $h(D,Q)$ which is more convenient in our problems. Let

$$\beta = \beta(D,Q) = \begin{cases} \sqrt{D/Q^2} & \text{if } D\geq \frac{1}{4}Q^2 \\ \frac{1}{2}(1+\sqrt{1-4D/Q^2}) & \text{if } D < \frac{1}{4}Q^2 \end{cases}.$$

Then

$$h(D,Q) = |\{q|D : \gcd(q, D/q) = 1, \beta Q < q \leq Q\}|. \tag{1}$$

In other words, $h(D,Q)$ counts certain divisors of D lying in a short interval (note that $\beta \ge \frac{1}{2}$). To study divisors of numbers in short intervals, Tenenbaum [11] introduced the functions

$$\tau(n,y,z) = \left|\{d|n : y < d \le z\}\right|,$$
$$H(x,y,z) = \left|\{n \le x : \tau(n,y,z) \ge 1\}\right|.$$

We will be more concerned with slight variations of these, namely

$$\tau^*(n,y,z) = \left|\{d|n : \gcd(d,n/d) = 1, y < d \le z\}\right|,$$
$$H^*(x,y,z) = \left|\{n \le x : \tau^*(n,y,z) \ge 1\}\right|.$$

Tenenbaum [11] (see also Theorem 21 of [6]) has determined the approximate growth of $H(x,y,z)$, and simple modifications of the proofs yield the following. For some constant c_1, if α, β are fixed with $0 < \alpha < \beta \le 1$, we have

$$H(Q^2, \alpha Q, \beta Q) \ll \frac{Q^2}{(\log Q)^{\theta}\sqrt{\log\log Q}}, \tag{2}$$

where $\theta = 1 - \frac{1+\log\log 2}{\log 2} = 0.08607\ldots$. If further $0 \le \gamma < \delta \le 1$ the following estimate holds:

$$H^*(\delta x^2, \alpha x, \beta x) - H^*(\gamma x^2, \alpha x, \beta x) \gg \frac{x^2}{(\log x)^{\theta}} \exp\left(-c_1\sqrt{\log\log x \log\log\log x}\right), \tag{3}$$

for $x \ge x_0(\alpha,\beta,\gamma,\delta)$.

Throughout, variables $c_1, c_2, \ldots$ denote positive absolute constants. Let $\omega(n)$ be the number of distinct prime factors of n and let $[x]$ be the greatest integer $\le x$.

3. Statements of Results

We raise two problems related to the `JC` problem for the Farey series. The first one is to estimate $M(Q)$ and determine the multiplicative structure of the numbers in Champs(Q). The second one is to estimate the quantities $H(Q)$, $H_r(Q)$ and $G_r(Q)$.

Theorem 1. *We have:*

$$M(Q) = \exp\left(2\log 2 \frac{\log Q}{\log\log Q} + O\left(\frac{\log Q}{(\log\log Q)^2}\right)\right).$$

Corollary 1. *If* $D \in \text{Champs}(Q)$ *then*

$$\omega(D) = 2\frac{\log Q}{\log\log Q} + O\left(\frac{\log Q}{(\log\log Q)^2}\right).$$

Since $\omega(n) \le 2\frac{\log Q}{\log\log Q} + O(\frac{\log Q}{(\log\log Q)^2})$ for all $n \le Q^2$, the JC have close to the maximum possible number of prime factors for integers of their size. In particular, most of the prime factors of $D \in \text{Champs}(Q)$ are small.

Corollary 2. *If* $\frac{R}{\log Q} \to \infty$ *as* $Q \to \infty$*, then almost all the prime factors of* D *are* $\le R$*.*

An interesting problem would be to bound the largest prime factor of any $D \in \text{Champs}(Q)$. For convenience in stating the next results, let

$$L(x) = \exp\left(c_1\sqrt{\log\log x \log\log\log x}\right). \tag{4}$$

Theorem 2. *We have:*

$$\frac{Q^2}{(\log Q)^\theta L(Q)} \ll H(Q) \ll \frac{Q^2}{(\log Q)^\theta \sqrt{\log\log Q}}.$$

We separate upper and lower bounds for $G_r(Q)$.

Theorem 3. *For any* $r \ge 2$*, we have*

$$G_r(Q) \ll Q^2\left(\frac{e\log 2(\log\log Q + c_2)}{\log r}\right)^{\frac{\log r}{\log 2}}.$$

Notice that the upper bound given by Theorem 3 is useless when $r \ll (\log Q)^{\xi e \log 2}$, where $\xi\log\xi = \theta/e$, since by Theorem 2 it follows that $G_r(Q) \le G_1(Q) \ll \frac{Q^2}{(\log Q)^\theta\sqrt{\log\log Q}}$.

We give two lower bounds for $G_r(Q)$. The first one is given by the following theorem.

Theorem 4. *Let* Q *be sufficiently large. We suppose* l *is an integer with* $1 \le l \le \frac{\log Q}{\log\log Q + c_3}$*, put* $r = \left[\frac{1}{l}\binom{2l}{l}\right]$*, and* $K = c_4(2\log\log Q + 2l)\log(2\log\log Q + 2l)$*. Then*

$$G_r(Q) \ge c_6\frac{Q^2}{(\log Q)^\theta L(Q)\binom{[2\log\log Q]+2l+2}{2l+2}K^{2l+2}}.$$

For example, the case $l=1$, $r=2$ gives

$$G_2(Q) \gg \frac{Q^2}{(\log Q)^\theta L(Q)} \cdot \frac{1}{(\log\log Q)^8 (\log\log\log Q)^4}.$$

By Theorems 2 and 4, if $l = o\left(\frac{\log\log Q}{\log\log\log Q}\right)$, equivalently $r = e^{o\left(\frac{\log\log Q}{\log\log\log Q}\right)}$, it follows that

$$G_r(Q) = \frac{Q^2}{(\log Q)^{\theta+o(1)}}.$$

On the other hand, when $\frac{l}{\log\log Q} \to \infty$, we get $G_r(Q) \ge \frac{Q^2}{l^{(2+o(1))l}}$. But since $l \sim \frac{\log r}{2\log 2}$, this yields $G_r(Q) \ge Q^2 (\log r)^{-(1+o(1))\frac{\log r}{\log 2}}$. This should be compared with the upper bound in Theorem 3.

A better lower bound of $G_r(Q)$ for intermediate l, $\log\log Q \ll l \le \frac{1}{2}\frac{\log Q}{\log\log Q}$, is given below.

Theorem 5. *Suppose* $2\log\log Q \le l \le \frac{1}{2}\frac{\log Q}{\log\log Q}$ *and put* $r = \frac{1}{11\log Q}\binom{2l}{l}$. *For positive absolute constants* c_6, c_7, *we have*

$$G_r(Q) \ge c_6 \frac{Q^2}{l^{5/2}\log^2 Q}\left(\frac{e}{2l}\left(\log\frac{\log Q}{l\log l} - c_7\frac{\log\log l}{\log l}\right)\right)^{2l}.$$

In this middle range for l, putting together Theorems 3 and 5, we obtain the following.

Corollary 3. *Suppose* $Q \ge Q_0$, $\frac{\log r}{\log\log Q \log\log\log Q} \to \infty$ *and* $\log r = (\log Q)^{o(1)}$. *Then*

$$G_r(Q) = Q^2 \left(\frac{(e\log 2 + o(1))\log\log Q}{\log r}\right)^{\frac{\log r}{\log 2}}.$$

An interesting problem would be to determine, if it exists, the limit

$$\lim_{Q\to\infty} \frac{H_r(Q)}{H(Q)}, \tag{5}$$

for each fixed r. The problem of comparing $H_r(x,y,z)$ to $H(x,y,z)$ (where $H_r(x,y,z) = |\{n \le x : \tau(n,y,z) = r\}|$) was studied by Tenenbaum [12], but the results are not strong enough to answer our question. In particular, Theorem 1 of [12] requires $z \le x^{1/(r+1)}$. The same question can be asked for the ratio $\frac{G_r(Q)}{H(Q)}$. The following argument supports the hypothesis that $G_r(Q) \gg_r H(Q)$. Suppose that for fixed α, β we have

$$H(x^2, \alpha x, \beta x) \asymp \frac{x^2}{(\log x)^\theta f(x)},$$

with $f(x)$ being smooth and slowly increasing. If $\tau\big(n, \frac{\sqrt{6}}{4}\frac{x}{\sqrt{6}}, \frac{\sqrt{6}}{3}\frac{x}{\sqrt{6}}\big) \geq 1$, then $\tau\big(6n, \frac{x}{2}, x\big) \geq 2$ and consequently

$$\sum_{i \geq 2} H_i\big(x^2, \tfrac{x}{2}, x\big) \gg \frac{x^2/6}{(\log x)^\theta f(x)} \gg H\big(x^2, \tfrac{x}{2}, x\big)\,.$$

4. Simple Inequalities

By (1), we immediately get the upper bound

$$h(D,Q) \leq \tau^*(D, Q/2, Q) \leq \tau(D, Q/2, Q). \tag{6}$$

A useful lower bound is

$$h(D,Q) \geq \begin{cases} \tau^*(D, 2Q/3, Q) & \text{if } Q^2/4 < D \leq 4Q^2/9 \\ 0 & \text{otherwise} \end{cases}. \tag{7}$$

A consequence of these bounds is

Lemma 1. *We have*

$$H(Q) \leq H^*(Q^2, Q/2, Q) \leq H(Q^2, Q/2, Q),$$

$$H_r(Q) \leq G_r(Q) \leq |\{D \leq Q^2 : \tau(D, Q/2, Q) \geq r\}|,$$

$$H(Q) \geq H^*(4Q^2/9, 2Q/3, Q) - H^*(Q^2/4, 2Q/3, Q),$$

$$G_r(Q) \geq \left|\{\tfrac{Q^2}{4} < D \leq \tfrac{4Q^2}{9} \ : \ \tau^*(D, \tfrac{2Q}{3}, Q) \geq r\}\right|.$$

Theorem 2 now follows immediately from (2), (3), and Lemma 1.

Suppose that $\omega(D) = k$. Then $h(D,Q) \leq \tau^*(D, Q/2, Q) \leq \tau^*(D, 0, D) = 2^k$. We can do a little bit better using Sperner's Theorem. Let $S = \{p^a : p^a \| D\}$. There is a natural bijection between the subsets of S and divisors q of D satisfying $\gcd(q, D/q) = 1$. Also, the relation $\subseteq$ defines a partial order on the subsets of S. In any chain of distinct subsets $R_1 \subseteq R_2 \subseteq \cdots \subseteq R_m$, at most one of the associated divisors is counted by $\tau^*(D, Q/2, Q)$. Sperner's Theorem (e.g. [10], page 732, fact # 7) tells us that the subsets of S can be partitioned into $\binom{k}{\lfloor k/2 \rfloor}$ chains. We thus have the following.

Lemma 2. *Let* $\omega(D)=k$. *Then*

$$h(D,Q)\le\binom{k}{\lfloor k/2\rfloor}\ll\frac{2^k}{\sqrt{k}}.$$

The inequality in Lemma 2 is nearly best possible, at least if k is not too large as a function of Q. For example, let k be even, let $p_1,\dots,p_{k-1}$ be distinct primes in the interval $\left(\left(\frac{2Q}{3}\right)^{2/k},\left(\frac{4Q}{5}\right)^{2/k}\right]$, let p_k be a prime in the interval $\left(\frac{9}{16}\left(\frac{2Q}{3}\right)^{2/k},\frac{9}{16}\left(\frac{4Q}{5}\right)^{2/k}\right]$, and set $D=p_1\cdots p_k$. Then $\frac14 Q^2<D\le\frac{9}{25}Q^2$. Let q be any product of $\frac{k}{2}$ primes p_i, $1\le i\le k-1$. Then $\frac23 Q<q<\frac45 Q$. By (7), q is counted in $h(D,Q)$, therefore

$$h(D,Q)\ge\binom{k-1}{k/2}=\frac12\binom{k}{k/2}\gg\frac{2^k}{\sqrt{k}}.$$

5. The Size of JC –Proof of Theorem 1

We turn to the problem of finding the size of $M(Q)$ when $Q\to\infty$.

Let q_j be the $j-$th prime and suppose $q_1\cdots q_m\le Q^2<q_1q_2\cdots q_{m+1}$. Then if $D\le Q^2$, it follows that $\omega(D)\le m$, and by Lemma 2, $M(Q)\le\binom{m}{[m/2]}\le 2^m$. Now $\theta(q_m)\le 2\log Q<\theta(q_{m+1})$, where θ is the Chebyshev function. By the Prime Number Theorem (with classical error term), we have $q_m=2\log Q+O\left((\log Q)e^{-c_8\sqrt{\log\log Q}}\right)$, which implies $m=\frac{2\log Q}{\log\log Q}+O\left(\frac{\log Q}{(\log\log Q)^2}\right)$. This gives the upper bound.

Let $A=\exp\left(c_9\frac{\log Q}{\log\log Q}\right)$, in which c_9 is some large constant. Suppose $q_1\cdots q_{2n}\le\frac{Q^2}{A^2}<q_1\cdots q_{2n+2}$. By the Prime Number Theorem, $n=\frac{\log Q}{\log\log Q}+O\left(\frac{\log Q}{(\log\log Q)^2}\right)$. If t is the product of any n primes $\le q_{2n}$, then $t=\frac{Q}{A}\exp\left(O\left(\frac{\log Q}{\log\log Q}\right)\right)$. Choose c_9 so that

$$\frac{Q}{A^{3/2}}\le q_1\cdots q_n\le t\le q_{n+1}\cdots q_{2n}\le\frac{Q}{A^{1/2}}.$$

By the box principle, for some $z\in\left[\frac{Q}{A^{3/2}},\frac{Q}{A^{1/2}}\right]$, the interval $[z,1.1z]$ contains $\gg\frac{1}{\log A}\binom{2n}{n}$ numbers t. Let s_1 be a prime in $\left(\frac{2Q/3}{z},\frac{0.7Q}{z}\right]$ and $D_0=q_1\cdots q_{2n}s_1$. Such an s_1 exists because $\frac{Q}{z}\ge A^{1/2}\to\infty$ as $Q\to\infty$. Then D_0 has $\gg\frac{1}{\log A}\binom{2n}{n}$ divisors in $(\frac23 Q,0.77Q]$ and $D_0\le\frac{Q^2}{A^2}\frac{0.7Q}{z}\le\frac{Q^2}{A^{1/2}}$.

Let $s_2 \neq s_1$, s_2 be a prime in $\left(\frac{Q^2/4}{D_0}, \frac{Q^2/3}{D_0}\right]$, and $D = s_2 D_0$. Such an s_2 exists because $\frac{Q^2}{D_0} \geq A^{1/2} \to \infty$ as $Q \to \infty$. Using (7), we conclude that

$$h(D,Q) \gg \frac{\binom{2n}{n}}{\log A} \gg \frac{2^{2n} \log\log Q}{\sqrt{n} \log Q} = \exp\left((2\log 2) \frac{\log Q}{\log\log Q} + O\left(\frac{\log Q}{(\log\log Q)^2} \right) \right).$$

This estimate concludes the proof of Theorem 1.

In order to prove Corollary 1, let us see that if D is a champion, then $h(D,Q) = M(Q) \leq 2^{\omega(D)}$ and therefore $\omega(D) \geq \frac{\log M(Q)}{\log 2}$. This gives the lower bound. The upper bound comes from the proof of Theorem 1.

6. The sizes of the prime factors of champions

Let now Q be large, and let $D \in \mathrm{Champs}(Q)$. Corollary 2 follows from the next lemma.

Lemma 3. *Suppose $R > 3\log Q$. Then the number of prime factors of D that are $\geq R$ is*

$$\ll \frac{\log Q}{\log\log Q} \frac{1}{\log\left(\frac{R}{3\log Q}\right)}.$$

In order to prove the lemma, let $\omega = \omega(D)$. Then by Corollary 1 and the Prime Number Theorem,

$$q_1 \cdots q_\omega \geq Q^2 \exp\left(-c_{10} \frac{\log Q}{\log\log Q} \right)$$

and $q_\omega \leq 3\log Q$ if Q is large. Since

$$Q^2 \geq D \geq q_1 \cdots q_{\omega-N} R^N \geq q_1 \cdots q_\omega \left(\frac{R}{q_\omega}\right) \geq Q^2 \exp\left(-c_{10} \frac{\log Q}{\log\log Q} \right) \left(\frac{R}{3\log Q} \right)^N,$$

we get the lemma.

Another corollary of Lemma 3 is that the champions are not too small.

Corollary 4. *If $D \in \mathrm{Champs}(Q)$, then $D \gg \frac{Q^2}{\log^3 Q}$.*

Proof. Assume $D \leq c_{11}/\log^3 Q$, where c_{11} is a sufficiently small positive constant. If c_{12} is large enough, then at most $\frac{\omega(D)}{10}$ prime factors of D are $\geq c_{12}\log Q$. Thus D has a divisor $d \in \left(\frac{Q}{2c_{12}^2 \log^2 Q}, \frac{Q}{2c_{12}\log Q} \right]$. Since $Q/d > 2c_{12}\log Q$, there exists a prime number $s \in \left(\frac{2}{3}\frac{Q}{d}, \frac{Q}{d}\right]$ which does not divide D. Since $Ds \leq \frac{Q^2}{\log Q}$, there exists a prime number $t \leq \frac{1}{4}\frac{Q^2}{Ds}$, which does not divide D. Let $D' = Dst \leq \frac{Q^2}{4}$. Then $\tau^*(D', \frac{2}{3}Q, Q) \geq M(Q)+1$, the divisors being the $M(Q)$ divisors of D which are in $(\beta(D,Q)Q,Q]$, plus sd. We obtained a contradiction, which completes the proof of the corollary. □

An interesting question which arises is the following. What is the approximate size range of $\frac{\tau^*(D,\beta Q,Q)}{\tau^*(D,0,D)}$ for champions D ? Is this ratio always $\gg (\log Q)^{-B}$ for some $B>0$?

7. Upper Bounds on $G_r(Q)$

In this section we prove Theorem 3.

Let l be the smallest integer with $\binom{l}{[l/2]} \ge r$. By Lemmas 1, 2 and a theorem of Hardy and Ramanujan [7], there is a constant c_2 so that

$$G_r(Q) \le \big|\{D \le Q^2 \,:\, \omega(D) \ge l\}\big| \ll \sum_{k\ge l} \frac{Q^2}{\log Q}\cdot\frac{(\log\log Q + c_2)^{k-1}}{(k-1)!}.$$

If $l-1 \le \log\log Q + c_2$, the bound in Theorem 3 is trivial, because the right hand side of (3) is $\ge Q^2$ in that case. Otherwise, we write $z=\log\log Q+c_2$, $l-1=\beta z$, and $\beta\ge 1$. Then

$$\frac{\log Q}{Q^2}G_r(Q) \ll \sum_{h\ge \beta z}\frac{z^h}{h!} \le \frac{1}{\beta^{\beta z}}\sum_{h\ge\beta z}\frac{(\beta z)^h}{h!} \le \left(\frac{e}{\beta}\right)^{\beta z} = \left(\frac{e(\log\log Q+c_2)}{l-1}\right)^{l-1}.$$

Lastly, $l-1 \le \frac{\log r}{\log 2}$ and the theorem follows.

8. Lower Bounds for $G_r(Q)$,

Proof of Theorem 4. First, the upper bound on l gives $K \le c_{13}\log Q$ and thus if $B = Q/K^{l+1}$, then

$$\begin{aligned} B \ge& Q\exp\left(-\left[1+\frac{\log Q}{\log\log Q + c_3}\right]\big(\log\log Q + \log c_{13}\big)\right)\\ \ge& \exp\left(c_{14}\frac{\log Q}{\log\log Q}\right)\end{aligned}$$

if c_3 is large enough. By (3), it follows that $H^*(B^2,B/2,B) \ge c_{15}\frac{B^2}{(\log B)^\theta L(B)}$.

Let

$$\tilde{H} = \big\{m\le B^2 \,:\, \tau^*(m,B/2,B)\ge 1,\ \omega(m)\le 2\log\log B\big\}.$$

By the Hardy-Ramanujan Theorem and Stirling's formula,

$$\begin{aligned}|\tilde{H}| \geq & c_{15}\frac{B^2}{(\log B)^\theta L(B)} - c_{16}\frac{B^2}{(\log B)^{2\log 2-1}} \geq c_{17}\frac{B^2}{(\log B)^\theta L(B)} \\ \geq & c_{17}\frac{B^2}{(\log B)^\theta L(Q)}.\end{aligned} \tag{8}$$

Let $m \in \tilde{H}$, and let $d \mid m$, $B/2 < d \leq B$, $\gcd\left(d, \frac{m}{d}\right) = 1$. Also let $\mathcal{J}$ be the interval

$$\mathcal{J} = \left(\frac{1}{c_{18}}\left(\frac{Q^2}{B^2}\right)^{\frac{1}{2l+2}}, \frac{1.1}{c_{18}}\left(\frac{Q^2}{B^2}\right)^{\frac{1}{2l+2}}\right].$$

By hypothesis, $\frac{1}{c_{18}}\left(\frac{Q^2}{B^2}\right)^{\frac{1}{2l+2}} \geq \frac{c_5}{c_{18}}(2\log\log Q + 2l)\log(2\log\log Q + 2l)$. By the Prime Number Theorem, if $\frac{c_5}{c_{18}}$ is sufficiently large, then $\mathcal{J}$ contains $\geq 2\log\log Q + 2l > 2\log\log B + 2l$ primes. Thus $\mathcal{J}$ contains primes $s_1, s_2, \ldots, s_{2l}$ that do not divide m. Then

$$D_0 := m s_1 s_2 \cdots s_{2l} \leq \left(\frac{1.1}{c_{18}}\right)^{2l} Q^2 K^{-2}.$$

Each product of l of the numbers s_i lies in the interval

$$I = \left(\left(\frac{1}{c_{18}}\right)^l \left(\frac{Q}{B}\right)^{\frac{2l}{2l+2}}, \left(\frac{1.1}{c_{18}}\right)^l \left(\frac{Q}{B}\right)^{\frac{2l}{2l+2}}\right].$$

By the box principle, for some $z \in \mathcal{I}$, the interval $(z, 1.1z]$ contains at least $\left[\frac{1}{l}\binom{2l}{l}\right] = r$ such products. Let t be a prime in $\left(\frac{2Q/3}{dz}, \frac{Q}{1.1dz}\right]$ that does not divide D_0. Since $\frac{Q}{dz} \geq \frac{Q}{Bz} \geq \frac{K^{l+1}}{(1.1K/c_{18})^l} = \left(\frac{c_{18}}{1.1}\right)^l K \geq K$ if $c_{18} \geq 1.1$, the interval contains $\geq 2\log\log B + 2l + 1$ primes, so t exists. Then tD_0 has at least r divisors q with $\gcd\left(q, \frac{tD_0}{q}\right) = 1$, and $2Q/3 < q \leq Q$. Let u be a prime in $\left(\frac{Q^2/4}{tD_0}, \frac{4Q^2/9}{tD_0}\right]$ that does not divide tD_0. Since

$$\frac{Q^2}{tD_0} \geq \left(\frac{c_{18}}{1.1}\right)^{2l} K^2 \frac{1.1dz}{Q} \geq 0.55\left(\frac{c_{18}}{1.21}\right)^l K^2 \frac{BK^l}{Q} > K$$

if $c_{18} \geq 3$, u exists. Finally let $D = utD_0 \in \left(\frac{1}{4}Q^2, \frac{4}{9}Q^2\right]$. By (7), $h(D,Q) \geq \frac{1}{l}\binom{2l}{l}$.

Since $D = uts_1 \cdots s_{2l}m$, at most $\binom{\omega(D)}{2l+2}$ values of m produce the same value of D. Since $\omega(D) \leq 2l + 2 + 2\log\log Q$, (7) implies the result.

Proof of Theorem 5.

Let c_{19} be large enough so that $\pi(c_{19}l\log l) \geq 20l$ for large l. Let $B = \left(\frac{Q}{c_{19}l\log l}\right)^{1/l}$. By hypothesis, $B > c_{19}l\log l$. Let $s_1 < \cdots < s_{2l}$ be any primes $\leq B$, and put $m = s_1 s_2 \cdots s_{2l}$.

There are $\binom{2l}{l}$ products $s_{i_1}\cdots s_{i_l}$, each lying in $[1,B^l]$. Thus, for some $z\in[1,B^l]$, the interval $(z,1.1z]$ contains at least r such products. Let q be a prime in $\left(\frac{2Q}{3z},\frac{Q}{1.1z}\right]$ that does not divide m. Such q exists because $Q/z\ge Q/B^l=c_{19}l\log l$ (i.e. $\pi\left(\frac{Q}{1.1z}\right)-\pi\left(\frac{2Q}{3z}\right)\ge 2l+2$). Let s be a prime in $\left(\frac{Q^2}{4qm},\frac{4Q^2}{9qm}\right]$ that does not divide qm. Such s exists because $1.1z\ge s_1\cdots s_l\ge\frac{m}{B^l}$ and thus $\frac{Q^2}{qm}\ge\frac{Qz}{1.1m}\ge\frac{1}{1.21}\frac{Q}{B^l}\ge\frac{c_{19}}{1.21}l\log l$. Put $D=sqm$, so $D\in\left(\frac{Q^2}{4},\frac{4Q^2}{9}\right]$. Also, $\tau^*(D,2Q/3,Q)\ge\tau^*(qm,2Q/3,Q)\ge r$, so $h(D,Q)\ge r$. Now each D comes from at most $\binom{2l+2}{2}$ values of m. Therefore,

$$\begin{aligned}G_r(Q)&\ge\frac{1}{\binom{2l+2}{2}}\sum_m\sum_{\substack{\frac{2q}{3z}<q\le\frac{Q}{1.1z}\\ \gcd(q,m)=1}}\sum_{\frac{Q^2}{4qm}<s\le\frac{4Q^2}{9qm}}1\\&\gg\frac{1}{l^2}\sum_m\sum_q\frac{Q^2/qm}{\log Q}\gg\frac{Q^2}{l^2\log^2 Q}\sum_m\frac{1}{m}.\end{aligned}$$

Lastly,

$$\begin{aligned}\sum_m\frac{1}{m}&\ge\frac{1}{(2l)!}\sum_{s_1\le B}\frac{1}{s_1}\sum_{\substack{s_2\le B\\ s_2\ne s_1}}\frac{1}{s_2}\cdots\sum_{\substack{s_{2l}\le B\\ s_{2l}\notin\{s_1,\dots,s_{2l-1}\}}}\frac{1}{s_{2l}}\\&\ge\frac{1}{(2l)!}\sum_{s_1\le B}\frac{1}{s_1}\sum_{q_2\le s_2\le B}\frac{1}{s_2}\cdots\sum_{q_{2l}\le s_{2l}\le B}\frac{1}{s_{2l}}\\&\ge\frac{1}{(2l)!}\left(\sum_{q_{2l}\le s\le B}\frac{1}{s}\right)^{2l}\\&\gg\left(\frac{e}{2l}\right)^{2l}\frac{1}{\sqrt{l}}\left(\log\log B-\log\log q_{2l}+O\left(\frac{1}{\log q_{2l}}\right)\right)^{2l}.\end{aligned}$$

Now

$$\log\log B=\log\frac{\log Q}{l}+O\left(\frac{\log l}{\log Q}\right),$$

$$\log\log q_{2l}=\log\log\left(2l(\log l+O(\log\log l))\right)=\log\log l+O\left(\frac{\log\log l}{\log l}\right)$$

and the theorem follows.

We conclude by proving Corollary 3. The upper bound comes from Theorem 3, while the lower bound follows from Theorem 5, by taking $l=\frac{\log r}{2\log 2}+O(\log\log Q)$.

9. Tables of Champions

Table 1: The H-values and the Champions for a given Q

Q	$\lvert\mathbf{D}(\mathcal{M}_\mathbf{Q})\rvert$	$\mathbf{H(Q)}$	$\mathbf{H_1(Q)}$	$\mathbf{H_2(Q)}$	$\mathbf{H_3(Q)}$	$\mathbf{H_4(Q)}$	$\mathbf{H_5(Q)}$	1/JC
2	1	1	1	0	0	0	0	2
3	2	2	2	0	0	0	0	3, 6
4	3	3	3	0	0	0	0	4, 6, 12
5	5	5	5	0	0	0	0	5, 10, 12, 15, 20
6	6	6	6	0	0	0	0	6, 10, 12, 15, 20, 30
7	9	9	9	0	0	0	0	7, 14, 15, 20, 21, 28, 30, 35, 42
8	11	11	11	0	0	0	0	8, 14, 20, 21, 24, 28, 30, 35, 40, 42, 56
180	4940	4481	4069	372	33	7	0	8190, 8580, 9240, 9570, 9660, 10010, 10710
181	5030	4594	4199	360	29	6	0	8580, 9240, 9570, 9660, 10010, 10710
182	5066	4608	4200	366	35	6	1	10010
183	5126	4654	4235	374	38	6	1	10010
184	5170	4686	4257	382	40	6	1	10010
185	5242	4744	4301	396	40	6	1	10010
186	5272	4754	4299	401	46	7	1	10010
187	5352	4825	4365	402	50	7	1	10010
188	5398	4860	4390	411	51	7	1	10010
189	5452	4905	4426	420	51	7	1	10010
190	5488	4918	4422	433	53	9	1	10010

Table 2: Selected lists of Champions

Champion	*Decomposition*	*No.ofappearances*	The values of Q
102	$2 \cdot 3 \cdot 17$	1	17
104	$2^3 \cdot 13$	2	13, 17

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Champion	*Decomposition*	*No.ofappearances*	The values of Q
105	$3 \cdot 5 \cdot 7$	2	17, 21
110	$2 \cdot 5 \cdot 11$	4	11–13, 17
112	$2^4 \cdot 7$	1	17
117	$3^2 \cdot 13$	2	13, 17
119	$7 \cdot 17$	1	17
120	$2^3 \cdot 3 \cdot 5$	1	17
126	$2 \cdot 3^2 \cdot 7$	6	17–22
130	$2 \cdot 5 \cdot 13$	2	13, 17
132	$2^2 \cdot 3 \cdot 11$	3	12, 13, 17
136	$2^3 \cdot 17$	1	17
143	$11 \cdot 13$	2	13, 17
144	$2^4 \cdot 3^2$	1	17
153	$3^2 \cdot 17$	1	17
154	$2 \cdot 7 \cdot 11$	4	17, 22–24
156	$2^2 \cdot 3 \cdot 13$	2	13, 17
165	$3 \cdot 5 \cdot 11$	1	17
168	$2^3 \cdot 3 \cdot 7$	5	24–28
170	$2 \cdot 5 \cdot 17$	1	17
176	$2^4 \cdot 11$	1	17
182	$2 \cdot 7 \cdot 13$	2	17, 26
187	$11 \cdot 17$	1	17
195	$3 \cdot 5 \cdot 13$	1	17
198	$2 \cdot 3^2 \cdot 11$	7	22–28
1540	$2^2 \cdot 5 \cdot 7 \cdot 11$	5	55, 59–62
1550	$2 \cdot 5^2 \cdot 31$	1	62
1566	$2 \cdot 3^3 \cdot 29$	4	59–62
1650	$2 \cdot 3 \cdot 5^2 \cdot 11$	2	75, 76
1674	$2 \cdot 3^3 \cdot 31$	1	62
1716	$2^2 \cdot 3 \cdot 11 \cdot 13$	5	55, 59–62
1798	$2 \cdot 29 \cdot 31$	1	62
1938	$2 \cdot 3 \cdot 17 \cdot 19$	4	59–62
1980	$2^2 \cdot 3^2 \cdot 5 \cdot 11$	5	55, 59–62
2310	$2 \cdot 3 \cdot 5 \cdot 7 \cdot 11$	27	70–96
2520	$2^3 \cdot 3^2 \cdot 5 \cdot 7$	5	72–76

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Champion	*Decomposition*	*No.ofappearances*	The values of Q
2730	$2 \cdot 3 \cdot 5 \cdot 7 \cdot 13$	18	91–108
3570	$2 \cdot 3 \cdot 5 \cdot 7 \cdot 17$	20	107–126
3990	$2 \cdot 3 \cdot 5 \cdot 7 \cdot 19$	15	114–126, 133–136
4290	$2 \cdot 3 \cdot 5 \cdot 11 \cdot 13$	1	130
4620	$2^2 \cdot 3 \cdot 5 \cdot 7 \cdot 11$	5	132–136
4830	$2 \cdot 3 \cdot 5 \cdot 7 \cdot 23$	1	138
5460	$2^2 \cdot 3 \cdot 5 \cdot 7 \cdot 13$	9	140–148
6930	$2 \cdot 3^2 \cdot 5 \cdot 7 \cdot 11$	43	126–168
7140	$2^2 \cdot 3 \cdot 5 \cdot 7 \cdot 17$	16	140–153, 167, 168
7590	$2 \cdot 3 \cdot 5 \cdot 11 \cdot 23$	12	167–178
7980	$2^2 \cdot 3 \cdot 5 \cdot 7 \cdot 19$	26	140–153, 167–178
8190	$2 \cdot 3^2 \cdot 5 \cdot 7 \cdot 13$	38	130–153, 167–180
8580	$2^2 \cdot 3 \cdot 5 \cdot 11 \cdot 13$	19	167–181, 195, 196, 201, 202
9240	$2^3 \cdot 3 \cdot 5 \cdot 7 \cdot 11$	14	168–181
9570	$2 \cdot 3 \cdot 5 \cdot 11 \cdot 29$	8	174–181
9660	$2^2 \cdot 3 \cdot 5 \cdot 7 \cdot 23$	15	167–181
20910	$2 \cdot 3 \cdot 5 \cdot 17 \cdot 41$	4	269–272
21930	$2 \cdot 3 \cdot 5 \cdot 17 \cdot 43$	4	269–272
22230	$2 \cdot 3^2 \cdot 5 \cdot 13 \cdot 19$	4	269–272
22440	$2^3 \cdot 3 \cdot 5 \cdot 11 \cdot 17$	4	269–272
22610	$2 \cdot 5 \cdot 7 \cdot 17 \cdot 19$	4	269–272
22770	$2 \cdot 3^2 \cdot 5 \cdot 11 \cdot 23$	4	269–272
23562	$2 \cdot 3^2 \cdot 7 \cdot 11 \cdot 17$	4	269–272
26334	$2 \cdot 3^2 \cdot 7 \cdot 11 \cdot 19$	4	269–272
27846	$2 \cdot 3^2 \cdot 7 \cdot 13 \cdot 17$	4	269–272
30030	$2 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 13$	87	231–233, 269–352
31122	$2 \cdot 3^2 \cdot 7 \cdot 13 \cdot 19$	8	269–272, 349–352
39270	$2 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 17$	56	269–272, 349–400
40698	$2 \cdot 3^2 \cdot 7 \cdot 17 \cdot 19$	8	349–356
43890	$2 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 19$	17	349–356, 399-407
53130	$2 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 23$	8	349–356

Table 3: The H-values for a given Q

Q	$\|D(\mathcal{M}_Q)\|$	$\frac{H_1(Q)}{H(Q)}$	$\frac{H_2}{H(Q)}$	$\frac{H_3(Q)}{H(Q)}$	$\frac{H_4(Q)}{H(Q)}$	$\frac{H_5}{H(Q)}$	$\frac{H_6(Q)}{H(Q)}$	$\frac{H_7(Q)}{H(Q)}$
31	154	0.9733	0.0267	0	0	0	0	0
32	162	0.9747	0.0253	0	0	0	0	0
33	172	0.9636	0.0303	0.0061	0	0	0	0
34	180	0.9408	0.0533	0.0059	0	0	0	0
35	192	0.956	0.033	0.011	0	0	0	0
100	1522	0.9059	0.0869	0.0065	0.0007	0	0	0
101	1572	0.9161	0.079	0.0042	0.0007	0	0	0
102	1588	0.9102	0.0829	0.0062	0.0007	0	0	0
103	1639	0.9233	0.0707	0.0053	0.0007	0	0	0
104	1663	0.9217	0.0718	0.0059	0.0007	0	0	0
105	1687	0.9156	0.0759	0.0071	0.0006	0.0006	0	0
106	1713	0.9108	0.0809	0.0071	0.0006	0.0006	0	0
107	1766	0.9212	0.0714	0.0062	0.0012	0	0	0
108	1784	0.9207	0.072	0.0061	0.0012	0	0	0
109	1838	0.9278	0.0658	0.0059	0.0006	0	0	0
110	1858	0.9216	0.0702	0.0076	0.0006	0	0	0
311	14770	0.89	0.0954	0.0121	0.0023	0.0001	0.0001	0
312	14818	0.8886	0.0965	0.0123	0.0025	0.0001	0.0001	0
313	14974	0.8927	0.0934	0.0117	0.0021	0.0001	0.0001	0
314	15052	0.8908	0.0954	0.0116	0.002	0.0001	0.0001	0
315	15124	0.8898	0.0959	0.0119	0.0022	0.0001	0.0001	0
316	15202	0.8892	0.0964	0.012	0.0022	0.0001	0.0001	0
317	15360	0.8931	0.0934	0.0113	0.002	0.0001	0.0001	0
318	15412	0.8914	0.0945	0.0118	0.0021	0.0001	0.0001	0
319	15552	0.8906	0.0954	0.0117	0.0022	0.0001	0.0001	0
320	15616	0.8903	0.0957	0.0117	0.0022	0.0001	0.0001	0

References

1. P. Erdős and E. G. Straus, *Remarks on the differences between consecutive primes*, Elem. Math. **35** (1980), 115–118.
2. P. X. Gallagher, *On the distribution of primes in short intervals*, Mathematika **23** (1976), 4–9.
3. P. X. Gallagher, *Corrigendum to distribution of primes in short intervals*, Mathematika **28** (1981), 86.

4. R. Harley, *Some estimates by Richard Brent applied to the "high jumpers" problem*, unpublished manuscript, available at ⟨http://pauillac.inria.fr/~harley/wnt.html⟩.

5. R. R. Hall, G. Tenenbaum, *On consecutive Farey arcs,* Acta. Arith. **44**, 1984, 397–405.

6. R. R. Hall, G. Tenenbaum, *Divisors,* Cambridge Tracts in mathematics vol. **90**, 1988.

7. G. H. Hardy, S. Ramanujan, *The normal number of prime factors of a number n*, Quart. J. Math. **158** (1917), 76–92.

8. H. Nelson: Problem 654, *J. Rec. Math.* **11**, (1978-79), 231.

9. Andrew Odlyzko, Michael Rubinstein, Marek Wolf *Jumping Champions,* Experiment. Math **8** (1999), no 2, 107–118.

10. K. Rosen, ed. *Handbook of Discrete and Combinatorial Mathematics*, CRC Press, 2000.

11. G. Tenenbaum, *Sur la probabilité qu'un entier possède un diviseur dans un intervalle donné*, Compositio Math. **51** (1984), 243–63.

12. G. Tenenbaum, *Un probabilité conditionnelle en arithmétique*, Acta Arith. **49** (1987), 165–187.

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