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On the distribution of imaginary parts of zeros of the Riemann zeta function, II

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Abstract. *Mathematics Subject Classification (2000):* Primary 11M26; Secondary 11K38 We continue our investigation of the distribution of the fractional parts of $\alpha\gamma$, where α is a fixed non-zero real number and γ runs over the imaginary parts of the non-trivial zeros of the Riemann zeta function. We establish some connections to Montgomery's pair correlation function and the distribution of primes in short intervals. We also discuss analogous results for a more general L-function.

Key words. Riemann zeta function – zeros, fractional parts – primes in short intervals – pair correlation functions

1. Introduction and Statement of Results

In this paper we continue the study of the distribution of the fractional parts $\{\alpha\gamma\}$ initiated by the first and third authors in [3], where α is a fixed positive real number and γ runs over the positive ordinates of zeros of the Riemann zeta function $\zeta(s)$. We extend and generalize the results from [3] in several directions, establishing connections between these fractional parts, the pair correlation of zeros of $\zeta(s)$ and the distribution of primes in short intervals. It is known [9] that for any fixed α, the fractional parts $\{\alpha\gamma\}$ are uniformly distributed (mod 1). That is, for all continuous

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functions $f : \mathbb{T} \to \mathbb{C}$, as $T \to \infty$ we have

$$\sum_{0<\gamma\le T} f(\alpha\gamma) = N(T)\int_{\mathbb{T}} f(x)dx + o(N(T)). \tag{1.1}$$

Here $\mathbb{T}$ is the torus $\mathbb{R}/\mathbb{Z}$ and $N(T)$ denotes the number of ordinates $0 < \gamma \le T$; it is well-known that

$$N(T) = \frac{T}{2\pi}\log\frac{T}{2\pi e} + O(\log T). \tag{1.2}$$

We are interested in the lower order terms in the asymptotic (1.1). For a general continuous function f the asymptotic (1.1) can be attained arbitrarily slowly so that no improvement of the error term there is possible. But if we assume that f has nice smoothness properties then we can isolate a second main term of size about T. More precisely, we define the function $g_\alpha : \mathbb{T} \to \mathbb{C}$ as follows. If α is not a rational multiple of $\frac{\log p}{2\pi}$ for some prime p, then g_α is identically zero. If $\alpha = \frac{a}{q}\frac{\log p}{2\pi}$ for some rational number a/q with $(a, q) = 1$ then we set

$$g_\alpha(x) = -\frac{\log p}{\pi}\Re\sum_{k=1}^{\infty}\frac{e^{-2\pi iqkx}}{p^{ak/2}} = -\frac{(p^{a/2}\cos 2\pi qx - 1)\log p}{\pi(p^a - 2p^{a/2}\cos 2\pi qx + 1)}. \tag{1.3}$$

Then, we expect (for suitable f) that as $T \to \infty$

$$\sum_{0<\gamma\le T} f(\alpha\gamma) = N(T)\int_{\mathbb{T}} f(x)dx + T\int_{\mathbb{T}} f(x)g_\alpha(x)dx + o(T). \tag{1.4}$$

As remarked above, certainly (1.4) does not hold for all continuous functions f. In Corollary 2 of [3], it is shown that (1.4) holds for all $f \in C^2(\mathbb{T})$, and if the Riemann Hypothesis (RH) is true then (1.4) holds for all absolutely continuous functions f (see Corollary 5 there). Moreover it is conjectured there (see Conjecture A there) that (1.4) does hold when f is the characteristic function of an interval in $\mathbb{T}$.

Conjecture 1. Let $\mathbb{I}$ be an interval of $\mathbb{T}$. Then

$$\sum_{\substack{0<\gamma\le T\\ \{\alpha\gamma\}\in\mathbb{I}}} 1 = |\mathbb{I}|N(T) + T\int_{\mathbb{I}} g_\alpha(x)dx + o(T),$$

uniformly in $\mathbb{I}$.

We define the discrepancy of the sequence $\{\alpha\gamma\}$ (for $0<\gamma\le T$) as

$$D_\alpha(T) = \sup_{\mathbb{I}} \Big| \frac{1}{N(T)} \sum_{\substack{0<\gamma\le T \\ \{\alpha\gamma\}\in\mathbb{I}}} 1 - |\mathbb{I}| \Big|,$$

where the supremum is over all intervals $\mathbb{I}$ of $\mathbb{T}$. Unconditionally, Fujii [4] proved that $D_\alpha(T) \ll \frac{\log\log T}{\log T}$ for every α. On RH, Hlawka [9] showed that $D_\alpha(T) \ll \frac{1}{\log T}$, which is best possible for α of the form $\frac{a}{q}\frac{\log p}{2\pi}$ ([3], Corollary 3). Conjecture 1 clearly implies the following conjecture for the discrepancy (see Conjecture A and Corollary 6 of [3]).

Conjecture 2. We have

$$D_\alpha(T) = \frac{T}{N(T)} \sup_{\mathbb{I}} \Big| \int_{\mathbb{I}} g_\alpha(x)dx \Big| + o\Big(\frac{1}{\log T}\Big).$$

Even assuming RH, we are unable to establish Conjectures 1 and 2. We show here some weaker results towards these conjectures, and how these conjectures would follow from certain natural assumptions on the zeros of $\zeta(s)$, or the distribution of prime numbers.

Theorem 1. *(i) We have unconditionally*

$$D_\alpha(T) \ge \frac{T}{N(T)} \sup_{\mathbb{I}} \Big| \int_{\mathbb{I}} g_\alpha(x)dx \Big| + o\Big(\frac{1}{\log T}\Big).$$

(ii) Assuming RH, for any interval $\mathbb{I}$ of $\mathbb{T}$ we have

$$\Big| \sum_{\substack{0<\gamma\le T \\ \{\alpha\gamma\}\in\mathbb{I}}} 1 - |\mathbb{I}|N(T) - T\int_{\mathbb{I}} g_\alpha(x)dx \Big| \le \frac{\alpha}{2}T + o(T).$$

The left side of (1.1) depends strongly on the behavior of the sums $\sum_{0<\gamma\le T} x^{i\gamma}$.

Conjecture 3. Let $A>1$ be a fixed real number. Uniformly for all $\frac{T^2}{(\log T)^5} \le x \le T^A$ we have

$$\sum_{0<\gamma\le T} x^{i\gamma} = o(T). \tag{1.5}$$

Theorem 2. *Assume RH. Then Conjecture 3 implies Conjectures 1 and 2.*

Remarks. Assuming RH, (1.5) holds for $x \to \infty$ and $x = o(T^2/\log^4 T)$ as $T \to \infty$ by uniform versions of Landau's formula for $\sum_{0<\gamma\le T} x^\rho$ [12]. For example, Lemma 1 of [3] implies, for $x > 1$ and $T \ge 2$, that (unconditionally)

$$\sum_{0<\gamma\le T} x^\rho = -\frac{\Lambda(n_x)}{2\pi}\frac{e^{iT\log(x/n_x)}-1}{i\log(x/n_x)} + O\left(x\log^2(Tx) + \frac{\log T}{\log x}\right), \quad (1.6)$$

where n_x is the nearest prime power to x, and the main term is to be interpreted as $-T\frac{\Lambda(x)}{2\pi}$ if $x = n_x$. This main term is always $\ll T\log x$. On RH, divide both sides of (1.6) by $x^{1/2}$ to obtain (1.5). Unconditionally, one can use Selberg's zero-density estimate to deduce

$$\Big|\sum_{0<\gamma\le T}(x^{i\gamma} - x^{\rho-1/2})\Big| \ll \frac{T\log^2(2x)}{\log T};$$

see e.g. (3.8) of [3]. This gives (1.5) when $\log x = o(\sqrt{\log T})$.

We next relate Conjecture 3 to the distribution of primes in short intervals.

Conjecture 4. For any $\varepsilon > 0$, if x is large and $y \le x^{1-\epsilon}$, then

$$\psi(x+y) - \psi(x) = y + o(x^{\frac12}/\log\log x).$$

Theorem 3. *Assume RH. Conjecture 4 implies Conjecture 3, and hence Conjectures 1 and 2. Conversely, if RH and Conjecture 3 holds, then for all fixed $\varepsilon > 0$, large x and $y \le x^{1-\varepsilon}$,*

$$\psi(x+y) - \psi(x) = y + o(x^{\frac12}\log x).$$

Remarks. Whereas the behavior of the left side of (1.6) is governed by a single prime when x is small, for larger x the sum is governed by the primes in an interval. It has been conjectured ([16], Conjecture 2) that for $x^\varepsilon \le h \le x^{1-\varepsilon}$, $\psi(x+h) - \psi(x) - h$ is normally distributed with mean 0 and variance $h\log(x/h)$. Thus, it is reasonable to conjecture that for every $\varepsilon > 0$,

$$\psi(x+y) - \psi(x) - y \ll_\varepsilon y^{1/2}x^\varepsilon \qquad (1 \le y \le x), \quad (1.7)$$

a far stronger assertion than Conjecture 4. It is known that RH implies $\psi(x) = x + O(x^{1/2}\log^2 x)$ (von Koch, 1900).

A statement similar to the second part of Theorem 3 has been given by Gonek ([7], Theorem 4). Assuming RH, Gonek showed that if

$$\sum_{0<\gamma\le T} x^{i\gamma} \ll_\varepsilon Tx^{-1/2+\varepsilon} + T^{1/2}x^\varepsilon$$

holds uniformly for all $x, T \geq 2$ and for each fixed $\varepsilon > 0$, then (1.7) follows.

We also want to describe how to bound the sum $\sum_{0<\gamma\leq T} x^{i\gamma}$ in terms of the pair correlation function

$$\mathcal{F}(x,T) = \sum_{0<\gamma,\gamma'\leq T} \frac{4x^{i(\gamma-\gamma')}}{4+(\gamma-\gamma')^2}. \tag{1.8}$$

Such bounds have been given by Gallagher and Mueller [5], Mueller [17], Heath-Brown [8], and Goldston and Heath-Brown [6]. First we state a strong version of the Pair Correlation Conjecture for $\zeta(s)$.

Conjecture 5. Fix a real number $A > 1$. Uniformly for all $\frac{T^2}{(\log T)^6} \leq x \leq T^A$ we have

$$\mathcal{F}(x,T) = N(T) + o\left(\frac{T}{\log T}\right) \qquad (T \to \infty).$$

Theorem 4. *Assume RH. Then Conjecture 5 implies Conjecture 3, and therefore also Conjectures 1 and 2.*

Remarks. The original pair correlation conjecture of Montgomery [14] states that

$$\mathcal{F}(x,T) \sim N(T) \qquad (T \to \infty)$$

uniformly for $T \leq x \leq T^A$, where A is any fixed real number. Tsz Ho Chan [1] has made an even stronger conjecture than Conjecture 5, namely he conjectured that for any $\epsilon > 0$ and any large $A > 1$,

$$\mathcal{F}(x,T) = N(T) + O\left(T^{1-\epsilon_1}\right)$$

if $T^{1+\epsilon} \leq x \leq T^A$, where $\epsilon_1 > 0$ may depend on ϵ, and the implicit constant may depend on ϵ and A.

In the next section, we prove Theorems 1–4. In section 3 we discuss analogous results for general L-functions.

2. Proof of Theorems 1–4

Proof of Theorem 1 (i). Let $\mathbb{I}$ denote an interval of $\mathbb{T}$ for which $|\int_{\mathbb{I}} g_\alpha(x)dx|$ attains its maximum. Let ϵ be a small positive number, and let $h_\epsilon : \mathbb{T} \to \mathbb{R}$ be a smooth function satisfying $h_\epsilon(x) \geq 0$ for all x, $h_\epsilon(x) = 0$ for $\epsilon < x \leq 1$, and $\int_{\mathbb{T}} h_\epsilon(x)dx = 1$. Set $f(x) = \int_{\mathbb{T}} h_\epsilon(y)\chi_{\mathbb{I}}(x-y)dy$, where $\chi_{\mathbb{I}}$

denotes the characteristic function of the interval $\mathbb{I}$. Then f is smooth, and so (1.4) holds for f. Therefore

$$\int_{\mathbb{T}} h_\epsilon(y)\Big(\sum_{\substack{0<\gamma\le T\\ \{\alpha\gamma\}\in\mathbb{I}+y}} 1 - N(T)|\mathbb{I}|\Big)\,dy = T\int_0^{\varepsilon} h_\epsilon(y)\int_{\mathbb{I}+y} g_\alpha(x)dx\,dy + o(T). \tag{2.1}$$

By (1.3), g_α is bounded and it follows that

$$\Big|\int_{\mathbb{I}+y} g_\alpha(x)dx - \int_{\mathbb{I}} g_\alpha(x)dx\Big| \ll \epsilon$$

for $0 \le y \le \varepsilon$. Therefore the right side of (2.1) equals

$$T\int_{\mathbb{I}} g_\alpha(x)dx + o(T) + O(\epsilon T).$$

It follows that for some choice of $y \in (0,\epsilon)$ one must have

$$\Big|\sum_{\substack{0<\gamma\le T\\ \{\alpha\gamma\}\in\mathbb{I}+y}} 1 - N(T)|\mathbb{I}|\Big| \ge T\Big|\int_{\mathbb{I}} g_\alpha(x)dx\Big| + o(T) + O(\epsilon T).$$

Letting $\epsilon \to 0$, we obtain our lower bound for the discrepancy.

Proof of Theorem 1 (ii) and Theorem 2. Let

$$h(u) = \begin{cases} 1 & \{u\}\in\mathbb{I}\\ 0 & \text{else}\end{cases}$$

and let J be a positive integer. There are trigonometric polynomials h^+ and h^-, depending on J and $\mathbb{I}$, satisfying

$$\begin{aligned} h^-(u) &\le h(u) \le h^+(u) \qquad (u\in\mathbb{R}),\\ h^{\pm}(u) &= \sum_{|j|\le J} c_j^{\pm} e^{2\pi i j u},\\ c_0^{\pm} &= |\mathbb{I}| \pm \frac{1}{J+1}, \qquad |c_j^{\pm}| \le \frac{1}{|j|} \quad (j\ge 1). \end{aligned}$$

For proofs, see Chapter 1 of [15], for example. These trigonometric polynomials are optimal in the sense that with J fixed, $|c_0^{\pm} - |\mathbb{I}||$ cannot be made smaller. We have

$$\sum_{0<\gamma\le T} h^-(\alpha\gamma) \le \sum_{\substack{0<\gamma\le T\\ \{\alpha\gamma\}\in\mathbb{I}}} 1 \le \sum_{0<\gamma\le T} h^+(\alpha\gamma).$$

For integers j, let $x_j = e^{2\pi j\alpha}$ and for positive j put

$$V_j = \frac{-\Lambda(n_{x_j})}{2\pi x_j^{1/2}} \frac{e^{iT\log(x_j/n_{x_j})} - 1}{i\log(x_j/n_{x_j})}.$$

Also define $V_{-j} = \overline{V_j}$. By (1.6), for nonzero j we have

$$\sum_{0<\gamma\le T} x_j^{i\gamma} = V_j + O\left(x_{|j|}^{1/2}\log^2(x_{|j|}T)\right).$$

This will be used for

$$1 \le |j| \le J_0 := \left\lfloor \frac{2\log T - 5\log\log T}{2\pi\alpha} \right\rfloor.$$

Suppose that $J \ge J_0$. We obtain (implied constants depend on α)

$$\begin{aligned}
\sum_{0<\gamma\le T} h^{\pm}(\alpha\gamma) &= c_0^{\pm} N(T) + \sum_{1\le|j|\le J} c_j^{\pm} \sum_{0<\gamma\le T} x_j^{i\gamma} \\
&= c_0^{\pm} N(T) + 2\Re \sum_{1\le j\le J_0} c_j^{\pm}\left[V_j + O(x_j^{1/2}\log^2 T)\right] \\
&\qquad + \sum_{J_0<|j|\le J} O\left(\frac{1}{|j|}\right)\Big|\sum_{0<\gamma\le T} x_j^{i\gamma}\Big| \\
&= |\mathbb{I}|N(T) + \sum_{j\ne 0} c_j^{\pm} V_j \pm \frac{N(T)}{J+1} + o(T) + \sum_{J_0<|j|\le J} O(|j|^{-1})\Big|\sum_{0<\gamma\le T} x_j^{i\gamma}\Big|,
\end{aligned}$$

where the term $o(T)$ is uniform in $\mathbb{I}$. If $\alpha = \frac{a}{q}\frac{\log p}{2\pi}$ for a prime p and coprime positive a, q, then $x_j = p^{aj/q}$ and consequently

$$V_{kq} = -\frac{T\log p}{2\pi p^{ak/2}}$$

for nonzero integers k. Thus,

$$\sum_{\substack{j\ne 0\\ q|j}} c_j^{\pm} V_j = T\int_{\mathbb{T}} h^{\pm} g_{\alpha}.$$

If $q \nmid j$, then x_j is not an integer. Hence

$$\sum_{\substack{j\ne 0\\ q\nmid j}} c_j^{\pm} V_j \ll T \sum_{\substack{1\le|j|\le J\\ q\nmid j}} \frac{1}{e^{\pi j\alpha}} \left|\frac{e^{iT\log(x_j/n_{x_j})} - 1}{iT\log(x_j/n_{x_j})}\right|.$$

The sum on the right converges uniformly in T, and each summand is $o(1)$ as $T \to \infty$, hence the left side is $o(T)$. We conclude

$$\sum_{j\neq 0} c_j^{\pm} V_j = T\int_{\mathbb{T}} h^{\pm} g_\alpha + o(T). \tag{2.2}$$

When α is not of the form $\frac{a}{q}\frac{\log p}{2\pi}$, x_j is never an integer (for nonzero j), and a similar argument yields (2.2). Since $h - h^{\pm}$ has constant sign,

$$\Big|\int_{\mathbb{T}} (h-h^{\pm}) g_\alpha\Big| \le \max_{x\in\mathbb{T}} |g_\alpha(x)| \int_{\mathbb{T}} |h-h^{\pm}| = \frac{\max_{x\in\mathbb{T}} |g_\alpha(x)|}{J+1} \ll \frac{1}{\log T}.$$

Therefore,

$$\sum_{0<\gamma\le T} h^{\pm}(\alpha\gamma) = |\mathbb{I}|N(T) + T\int_{\mathbb{T}} h g_\alpha + o(T) \pm \frac{N(T)}{J+1} + \sum_{J_0<|j|\le J} O\left(\frac{1}{|j|}\right)\Big|\sum_{0<\gamma\le T} x_j^{i\gamma}\Big|.$$

For Theorem 1 (ii), we take $J = J_0$. For Theorem 2, take $J = \lfloor \lambda \log T\rfloor$ with λ fixed, and then let $\lambda \to \infty$.

Proof of Theorem 3. We first construct a function F which is a good approximation of the characteristic function of the interval $[0,1]$ and whose Fourier transform is supported on $[-K,K]$, where K is a parameter to be specified later. Consider the entire function

$$H(z) = \left(\frac{\sin \pi z}{\pi}\right)^2 \left(\sum_{n=1}^{\infty} \frac{1}{(z-n)^2} - \sum_{n=1}^{\infty} \frac{1}{(z+n)^2} + \frac{2}{z}\right)$$

for complex z, and set

$$F(z) = \frac{H(Kz) + H(K-Kz)}{2}.$$

The function $H(z)$ is related to the so-called Beurling-Selberg functions, and basic facts about H can be found in [23]. In particular, for real x, (i) $H(x)$ is an odd function; (ii) the Fourier transform $\widehat{H}$ is supported on $[-1,1]$; (iii) $H(x) = \operatorname{sgn}(x) + O(\frac{1}{1+|x|^3})$, where $\operatorname{sgn}(x)=1$ if $x>0$, $\operatorname{sgn}(x) = -1$ if $x<0$ and $\operatorname{sgn}(0)=0$; (iv) $H'(x) = O(\frac{1}{1+|x|^3})$. Item (iii) follows from (2.26) of [23] and the Euler-Maclaurin summation formula, and (iv) follows from Theorem 6 of [23]. Let I be the indicator function of the interval $[0,1]$. It follows that the Fourier transform $\widehat{F}$ of F is supported on $[-K,K]$ and

$$|F(x)-I(x)| \ll \frac{1}{1+K^3|x|^3}+\frac{1}{1+K^3|1-x|^3}. \tag{2.3}$$

Since

$$\widehat{I}(t)=\frac{1-e^{-2\pi it}}{2\pi it},$$

it follows readily that $\widehat{F}(t) \ll 1$, uniformly in K, and

$$\begin{aligned}\widehat{F}'(t) &= \frac{1-(1+2\pi it)e^{-2\pi it}}{-2\pi it^2}+O\Big(\int_{-\infty}^{\infty}\frac{|x|}{1+K^3|x|^3}+\frac{|x|}{1+K^3|1-x|^3}\,dx\Big)\\ &= O\left(\frac{1}{1+|t|}+\frac{1}{K}\right).\end{aligned}$$

Next, let $T \ge 2$ and $T \le x \le T^A$. Write

$$\sum_{0<\gamma\le T} x^{i\gamma} = \sum_{|\gamma|\le x} x^{i\gamma}F(\gamma/T)+\sum_{|\gamma|\le x} x^{i\gamma}\big[I(\gamma/T)-F(\gamma/T)\big].$$

By (1.2) and (2.3), the second sum on the right is

$$\begin{aligned}&\ll N\left(\frac{T}{K}\right)+\left(N\left(T+\frac{T}{K}\right)-N\left(T-\frac{T}{K}\right)\right)\\ &\quad+\frac{T^3}{K^3}\Bigg(\sum_{|\gamma|>T/K}\frac{1}{|\gamma|^3}+\sum_{|\gamma-T|\ge T/K}\frac{1}{|\gamma-T|^3}\Bigg)\\ &\ll \frac{T\log T}{K}.\end{aligned}$$

Also,

$$\begin{aligned}\sum_{|\gamma|\le x} x^{i\gamma}F(\gamma/T) & \sum_{|\gamma|\le x} x^{i\gamma}\int_{-K}^{K} e^{2\pi iv\gamma/T}\widehat{F}(v)\,dv\\ &= x^{-1/2}\int_{-K}^{K} e^{-\pi v/T}\widehat{F}(v)\sum_{|\gamma|\le x}\left(xe^{2\pi v/T}\right)^{\rho}\,dv\\ &= -\frac{T}{2\pi x^{1/2}}\int_{-K}^{K} e^{-\pi v/T}\left(\widehat{F}'(v)-\frac{\pi}{T}\widehat{F}(v)\right)\sum_{|\gamma|\le x}\frac{\left(xe^{2\pi v/T}\right)^{\rho}}{\rho}\,dv,\end{aligned}$$

where the last line follows from the previous line using integration by parts. The final sum on γ is evaluated using the explicit formula (see e.g. [2], §17)

$$G(x) := \psi(x) - x = -\sum_{|\gamma|\le M} \frac{x^\rho}{\rho} + O\left(\log x + \frac{x\log^2(Mx)}{M}\right), \tag{2.4}$$

valid for $x \ge 2$, $M \ge 2$. Since

$$\int_{-K}^{K} e^{-\pi v/T}\left(\widehat{F}'(v) - \frac{\pi}{T}\widehat{F}(v)\right)\,dv = 0,$$

we obtain

$$\sum_{|\gamma|\le x} x^{i\gamma}F(\gamma/T) = \frac{-T}{2\pi\sqrt{x}}\int_{-K}^{K}\widehat{F}'(v)\left(G(xe^{2\pi v/T}) - G(x)\right)\,dv \\ + O\left(K\left(1+Tx^{-1/2}\right)\log^2 x\right).$$

Altogether, this gives

$$\sum_{|\gamma|\le T} x^{i\gamma} \ll \frac{T\log K}{\sqrt{x}} \max_{xe^{-2\pi K/T}\le y\le xe^{2\pi K/T}} |G(y)-G(x)| \\ + \frac{T\log T}{K} + K\left(1+Tx^{-1/2}\right)\log^2 x.$$

Take $K = \log^2 T$ and assume Conjecture 4. The first part of Theorem 3 follows.

The second part is straightforward, starting with the explicit formula (2.4) in the form

$$\psi(x+y) - \psi(x) - y = -\sum_{|\gamma|\le x}\frac{(x+y)^\rho - x^\rho}{\rho} + O(\log^2 x).$$

Fix $\varepsilon > 0$ and apply Conjecture 3 with $A = 2/\varepsilon$. By partial summation,

$$\begin{aligned}\Big|\sum_{x^{\varepsilon/2}<|\gamma|\le x}\frac{x^\rho}{\rho}\Big| &= 2\Big|\Re\sum_{x^{\varepsilon/2}<\gamma\le x}\frac{x^\rho}{\rho}\Big| \\ &\le 2x^{1/2}\Big|\frac{1}{\frac12+ix}\sum_{0<\gamma\le x}x^{i\gamma} + i\int_{x^{\varepsilon/2}}^{x}\frac{1}{(\frac12+it)^2}\sum_{0<\gamma\le t}x^{i\gamma}\,dt\Big| \\ &= o(x^{1/2}\log x).\end{aligned}$$

The smaller zeros are handled in a trivial way. We have, for $y \le x$,

$$(x+y)^\rho - x^\rho = x^\rho \left(\rho \frac{y}{x} + O\left(\frac{|\rho|^2 y^2}{x^2} \right) \right),$$

whence

$$\sum_{|\gamma| \le x^{\varepsilon/2}} \frac{(x+y)^\rho - x^\rho}{\rho} \ll N(x^{\varepsilon/2}) x^{1/2} \left(\frac{y}{x} + x^{\varepsilon/2} \frac{y^2}{x^2} \right) \ll x^{\frac{1}{2} - \frac{\varepsilon}{2}} \log x.$$

Therefore, $\psi(x+y) - \psi(x) - y = o(x^{1/2} \log x)$, as claimed.

Proof of Theorem 4. It will be convenient to work with the normalized sum

$$\mathcal{D}(x,T) = \frac{\mathcal{F}(x,T)}{N(T)}.$$

Lemma 1. *Suppose $T \ge 10$ and $1 \le \beta \le \frac{T}{2 \log T}$. Then*

$$\begin{aligned}
\sum_{0<\gamma\le T} x^{i\gamma} &\ll T \left(\frac{\log T}{\beta} \right)^{\frac{1}{2}} \Bigg(1 + \max_{\frac{T}{\beta \log T} \le t \le T} |\mathcal{D}(x,t)| \\
&\qquad + \beta^3 \left| \int_{-\infty}^{\infty} (\mathcal{D}(xe^u, t) - \mathcal{D}(x,t)) e^{-2\beta|u|} \, du \right| \Bigg)^{\frac{1}{2}} \\
&\ll \frac{T (\log T)^{\frac{1}{2}}}{\beta^{1/2}} \left(1 + \max_{\frac{T}{\beta \log T} \le t \le T} |\mathcal{D}(x,t)| \right)^{1/2} + T (\beta \log T)^{1/2} \\
&\quad \times \left(\max_{\frac{T}{\beta \log T} \le t \le T} \ \max_{0 \le u \le \frac{1}{\beta} \log(\beta \log T)} |\mathcal{D}(xe^u, t) + \mathcal{D}(xe^{-u}, t) - 2\mathcal{D}(x,t)| \right)^{\frac{1}{2}}.
\end{aligned}$$

Proof. We follow [6] by estimating $\sum_{0<\gamma\le T} x^{i\gamma}$ in terms of

$$G_\beta(x,T) = \sum_{0<\gamma,\gamma'\le T} \frac{4\beta^2 x^{i(\gamma-\gamma')}}{4\beta^2 + (\gamma-\gamma')^2}.$$

In particular, $G_1(x,T) = \mathcal{F}(x,T)$, and by (1.2), we have $G_\beta(x,T) \ll (1+\beta) T \log^2 T$. By Lemma 1 of [6], uniformly for $1 \le \beta \le T$ and $1 \le V \le T$, we have

$$\begin{aligned}
\sum_{0<\gamma\le T} x^{i\gamma} &\ll \left(T\beta^{-1} \max_{t \le T} G_\beta(x,t) \right)^{1/2} \\
&\ll \frac{T \log T}{V^{1/2}} + \left(T\beta^{-1} \max_{T/V \le t \le T} G_\beta(x,t) \right)^{1/2}.
\end{aligned} \tag{2.5}$$

Using Lemma 2 of [6], we have

$$
\begin{aligned}
G_\beta(x,t) &= \beta^2 \mathcal{F}(x,t) + \beta(1-\beta^2)\int_{-\infty}^{\infty} \mathcal{F}(xe^u,t)e^{-2\beta|u|}\,du \\
&= \mathcal{F}(x,t) + \beta(1-\beta^2)\int_{-\infty}^{\infty} (\mathcal{F}(xe^u,t)-\mathcal{F}(x,t))e^{-2\beta|u|}\,du,
\end{aligned}
$$

from which the first inequality in the lemma follows upon taking $V = \beta \log T$. For the second inequality, combine the terms in the integral with $u = v$ and $u = -v$ for $0 \le v \le \frac{\log(\beta \log T)}{\beta}$, and use the trivial bound $\mathcal{D}(z,t) \ll \log t$ when $|u| \ge \frac{\log(\beta\log T)}{\beta}$ ($z = x$ and $z = xe^u$).

In order to finish the proof of Theorem 4, suppose that $\log T \le \beta \le \log^2 T$. From Conjecture 5 it follows that the terms $\mathcal{D}(xe^u,t)$, $\mathcal{D}(xe^{-u},t)$, and $\mathcal{D}(x,t)$, in the ranges from the statement of the above lemma, are all of the form $1 + o\big((\log T)^{-2}\big)$. Therefore,

$$
\sum_{0<\gamma\le T} x^{i\gamma} = O\left(T\frac{(\log T)^{1/2}}{\beta^{1/2}}\right) + o\left(T\frac{\beta^{1/2}}{(\log T)^{1/2}}\right).
$$

Thus, taking β slightly larger than $\log T$ produces the desired result.

3. General *L*-functions

Consider a Dirichlet series $F(s) = \sum_{n=1}^{\infty} a_F(n)n^{-s}$ satisfying the following axioms:
(i) there exists an integer $m \ge 0$ such that $(s-1)^m F(s)$ is an entire function of finite order;
(ii) F satisfies a functional equation of the type:

$$
\Phi(s) = w\overline{\Phi}(1-s),
$$

where

$$
\Phi(s) = Q^s \prod_{j=1}^{r} \Gamma(\lambda_j s + \mu_j) F(s)
$$

with $Q > 0$, $\lambda_j > 0$, $\Re(\mu_j) \ge 0$ and $|w| = 1$. (Here, $\overline{f}(s) = \overline{f(\overline{s})}$);
(iii) $F(s)$ has an Euler product, which we write as

$$
-\frac{F'}{F}(s) = \sum_{n=1}^{\infty} \Lambda_F(n) n^{-s},
$$

where $\Lambda_F(n)$ is supported on powers of primes.

We also need some growth conditions on the coefficients $a_F(n)$ and $\Lambda_F(n)$. Although stronger than we require, for convenience we impose the conditions (iv) $\Lambda_F(n) \ll n^{\theta_F}$ for some $\theta_F < \frac{1}{2}$ and (v) for every $\varepsilon > 0$, $a_F(n) \ll_\varepsilon n^\varepsilon$. Together, conditions (i)–(v) define the *Selberg class* $\mathcal{S}$ of Dirichlet series. For a survey of results and conjectures concerning the Selberg class, the reader may consult Kaczorowski and Perelli's paper [10]. In particular, $\mathcal{S}$ includes the Riemann zeta function, Dirichlet L-functions, and L-functions attached to number fields and elliptic curves. The Selberg class is conjectured to equal the class of all automorphic L-functions, suitably normalized so that their nontrivial zeros have real parts between 0 and 1.

The functional equation is not uniquely determined in light of the duplication formula for Γ-function, however the real sum

$$d_F = 2\sum_{j=1}^{r} \lambda_j$$

is well-defined and is known as the degree of F. Analogous to (1.2), we have (cf. [22], (1.6))

$$\begin{aligned} N_F(T) &= |\{\rho = \beta + i\gamma : F(\rho) = 0, 0 < \beta < 1, 0 < \gamma \le T\}| \\ &= \frac{d_F}{2\pi} T \log T + c_1 T + O(\log T) \end{aligned} \tag{3.1}$$

for some constant $c_1 = c_1(F)$. A function $F \in \mathcal{S}$ is said to be *primitive* if it cannot be written as a product of two or more elements of $\mathcal{S}$. We henceforth assume that F is primitive. The extension of our results to non-primitive F is straightforward. It is expected that all zeros of F with real part between 0 and 1 have real part $\frac{1}{2}$, a hypothesis we abbreviate as RH_F. Although we shall assume RH_F for many of the results in this section, sometimes a weaker hypothesis suffices, that most zeros of F are close to the critical line.

Hypothesis Z_F. There exist constants $A > 0, B > 0$ (depending on F) such that

$$\begin{aligned} N_F(\sigma, T) &= \left| \left\{ \beta + i\gamma : \frac{1}{2} \le \beta \le \sigma, 0 < \gamma \le T \right\} \right| \\ &\ll T^{1 - A(\sigma - 1/2)} \log^B T, \end{aligned}$$

uniformly for $\sigma \ge 1/2$ and $T \ge 2$.

Hypothesis Z_F is known, with $B = 1$, for the Riemann zeta function and Dirichlet L-functions (Selberg [20], [21]), and certain degree 2 L-functions attached to cusp forms (Luo [13]).

The next tool we require is an analog of (1.6). It is very similar to Proposition 1 of [19], and with small modifications to that proof we obtain the following result, which is nontrivial provided $x^{1/2+\theta_F} + x^{1/2+\varepsilon} \ll T$.

Lemma 2. *Let $F \in \mathcal{S}$, $x > 1$, $T \geq 2$, and let n_x be a nearest integer to x. Then, for any $\varepsilon > 0$,*

$$\sum_{0<\gamma\leq T} x^\rho = -\frac{\Lambda_F(n_x)}{2\pi}\frac{e^{iT\log(x/n_x)}-1}{i\log(x/n_x)} + O_\varepsilon\left(x^{1+\theta_F}\log(2x) + x^{1+\varepsilon}\log T + \frac{\log T}{\log x}\right).$$

Using Lemma 2 in place of Lemma 1 of [3], Hypothesis Z_F in place of Lemma 2 of [3], and following the proof of Theorem 1 of [3], we obtain a generalization of (1.4).

Theorem 5. *Let $F \in \mathcal{S}$. If $\alpha = \frac{a\log p}{2\pi q}$ for some prime number p and positive integers a, q with $(a,q) = 1$, define*

$$g_{F,\alpha}(t) = -\frac{1}{\pi}\Re\sum_{k=1}^{\infty}\frac{\Lambda_F(p^{ak})}{p^{ak/2}}e^{-2\pi iqkt}.$$

For other α, define $g_{F,\alpha}(t) = 0$ for all t. If Hypothesis Z_F holds, then

$$\sum_{0<\gamma\leq T} f(\alpha\gamma) = N_F(T)\int_{\mathbb{T}} f(x)\,dx + T\int_{\mathbb{T}} f(x)g_{F,\alpha}(x)\,dx + o(T) \qquad (3.2)$$

for all $f \in C^2(\mathbb{T})$. Assuming RH_F, (3.2) holds for all absolutely continuous f.

Since Hypothesis Z_F holds for Dirichlet L-functions $L(s,\chi)$, we obtain the following.

Corollary 1. *Unconditionally, for Dirichlet L-functions F, (3.2) holds for all $f \in C^2(\mathbb{T})$.*

When $F(s) = L(s,\chi)$ and $\alpha = \frac{a\log p}{2\pi q}$ with p prime, $(a,q) = 1$, we have

$$g_{F,\alpha}(t) = -\frac{\log p}{\pi}\Re\left(\frac{e^{2\pi i(qt+a\xi)}}{p^{a/2} - e^{2\pi i(qt+a\xi)}}\right),$$

where $\chi(p) = e^{2\pi i\xi}$. It follows that there is a shortage of zeros of $L(s,\chi)$ with $\{\alpha\gamma\}$ near $\frac{k-a\xi}{q}$, $k = 0, \cdots, q-1$. We illustrate this phenomenon with three histograms of $M_F(y;T)$, where

$$M_F(y) = \frac{T}{N_F(T)}\Bigg|\sum_{\substack{0<\gamma\le T\\ \{\alpha\gamma\}<y}} 1 - yN_F(T)\Bigg|,$$

F a Dirichlet L-function associated with a character of conductor 5 and $T = 500,000$. For both characters, $N_F(T) = 946488$. The list of zeros was taken from Michael Rubinstein's data files on his Web page. In Figure 1 we plot for each subinterval $I = [y, y + \frac{1}{500})$ the value of $500(M_F(y + \frac{1}{500}) - M_F(y))$ and also the graph of $g_{F,\alpha}(y)$. The characters are identified by their value at 2.

We conjecture that (3.2) holds when f is the indicator function of an interval, and are thus led to the following generalizations of Conjectures 1 and 2. Here $D_{F,\alpha}$ is the natural generalization of the discrepancy function D_α.

Conjecture 6. Let $\mathbb{I}$ be an interval of $\mathbb{T}$. Then

$$\sum_{\substack{0<\gamma\le T\\ \{\alpha\gamma\}\in\mathbb{I}}} 1 = |\mathbb{I}|N_F(T) + T\int_{\mathbb{I}} g_{F,\alpha}(x)dx + o(T).$$

Conjecture 7. We have

$$D_{F,\alpha}(T) = \frac{T}{N_F(T)}\sup_{\mathbb{I}}\Big|\int_{\mathbb{I}} g_{F,\alpha}(x)\,dx\Big| + o\Big(\frac{1}{\log T}\Big).$$

Combining Theorem 5 and the proof of Theorem 1, we obtain the following. The only difference in the proof is that here we take

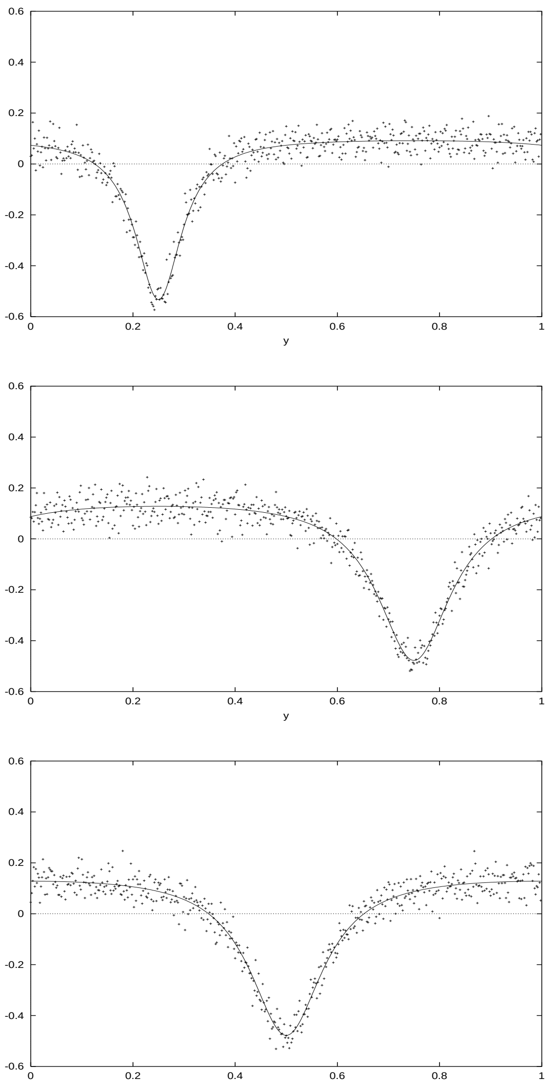
$$J_0 = \left\lfloor \frac{\frac{\log T}{1/2+\theta_F} - 5\log\log T}{2\pi\alpha} \right\rfloor.$$

Theorem 6. *(i) Assuming Hypothesis Z_F, we have*

$$D_{F,\alpha}(T) \ge \frac{T}{N_F(T)}\sup_{\mathbb{I}}\Big|\int_{\mathbb{I}} g_{F,\alpha}(x)\,dx\Big| + o\Big(\frac{1}{\log T}\Big).$$

(ii) Assuming RH_F, for any interval $\mathbb{I}$ of $\mathbb{T}$ we have

$$\Big|\sum_{\substack{0<\gamma\le T\\ \{\alpha\gamma\}\in\mathbb{I}}} 1 - |\mathbb{I}|N_F(T) - T\int_{\mathbb{I}} g_{F,\alpha}(x)dx\Big| \le \alpha(1/2+\theta_F)T + o(T).$$



$\alpha = \frac{\log 2}{2\pi}, \chi(2) = -i$

$\alpha = \frac{\log 3}{2\pi}, \chi(2) = -i$

$\alpha = \frac{\log 3}{2\pi}, \chi(2) = -1$

Fig. 1. $500(M_F(y + \frac{1}{500}) - M_F(y))$ vs. $g_{F,\alpha}(y)$ for $T = 500000$.

We can prove a direct analog of Theorem 2, by requiring a slightly larger range of T in the analog of Conjecture 3, since θ_F may be large.

Conjecture 8. Let $A > 1$ be a fixed real number. Uniformly for

$$\frac{T^{1/(1/2+\theta_F)}}{\log^5 T} \le x \le T^A,$$

we conjecture that

$$\sum_{0<\gamma\le T} x^{i\gamma} = o(T). \tag{3.3}$$

Theorem 7. *Assume RH_F. Then Conjecture 8 implies Conjectures 6 and 7.*

The analog of Theorem 3 holds for $F \in \mathcal{S}$, by following the proof given in the preceding section. Here we need an explicit formula similar to (2.4). By standard contour integration methods, one obtains

$$G_F(x) := \sum_{n\le x} \Lambda_F(n) - d_F x = -\sum_{|\rho|\le Q} \frac{x^\rho}{\rho} + O(x^{\theta_F}\log x)$$

provided $Q \ge x\log x$. Since $\theta_F < \frac{1}{2}$, the error term is acceptable.

Conjecture 9. For every $\varepsilon > 0$, if x is large and $y \le x^{1-\epsilon}$, then

$$G_F(x+y) - G_F(x) = o(x^{\frac{1}{2}}/\log\log x).$$

Theorem 8. *Assume RH_F. Conjecture 9 implies Conjecture 8, and hence Conjectures 6 and 7. Conversely, if RH_F and Conjecture 8 holds, then for all fixed $\varepsilon > 0$, large x and $y \le x^{1-\varepsilon}$,*

$$G_F(x+y) - G_F(x) = o(x^{\frac{1}{2}}\log x).$$

In order to address an analog of Theorem 4, we first quote a Pair Correlation Conjecture for F, due to Murty and Perelli [18].

Conjecture 10. Define

$$\mathcal{F}_F(x,T) = \sum_{0<\gamma,\gamma'\le T} \frac{4x^{i(\gamma-\gamma')}}{4+(\gamma-\gamma')^2}$$

and $\mathcal{D}_F(x,T) = \mathcal{F}_F(x,T)/N_F(T)$. We have $\mathcal{D}_F(T^{\theta d_F},T) \sim \theta$ for $0 < \theta \le 1$ and $\mathcal{D}(T^{\theta d_F},T) \sim 1$ for $\theta \ge 1$.

Notice that, as a function of x, $\mathcal{F}_F(x,T)$ is conjectured to undergo a change of behavior in the vicinity of $x = T^{d_F}$. In order to deduce Conjecture 8, we can postulate a stronger version of Conjecture 10, with error terms of relative order $o(1/\log^2 T)$. We succeed, as in the proof of Theorem 4, when $d_F = 1$. When $d_F \geq 2$, however, this transition zone lies outside the range in which Lemma 2 is useful (Kaczorowski and Perelli recently proved that $1 < d_F < 2$ is impossible [11]; it is conjectured that d_F is always an integer). We can use an analog of Lemma 2, which follows by the same method (replace $\mathcal{D}(x,T)$ with $\mathcal{D}_F(x,T)$). However, in order to prove the right side is small, we require that $\mathcal{D}_F(x,T)$ has small *variation*, even through the transition zone $x \approx T^{d_F}$. Tsz Ho Chan [1] studied the behavior of $\mathcal{D}(x,T)$ (for $\zeta(s)$) in the vicinity of $x = T$ assuming RH plus a quantitative version of the twin prime conjecture with strong error term. His analysis leads to a pair correlation conjecture with $\mathcal{D}(x,T)$ smoothly varying through the transition zone. We conjecture that the same holds for other $F \in \mathcal{S}$.

Conjecture 11. For $F \in \mathcal{S}$, $\mathcal{D}_F(x,T) \ll 1$ uniformly in x and T, and for any $A > 0$ there is a $c > 0$ so that

$$|\mathcal{D}_F(x+\delta x,T) + \mathcal{D}_F(x-\delta x,T) - 2\mathcal{D}_F(x,T)| = o(T/\log T)$$

uniformly for $T \leq x \leq T^A$ and $0 \leq \delta \leq (\log T)^{c-1}$.

Following the proof of Theorem 4 (take $\beta = \log T \log\log T$, for example), we arrive at the following.

Theorem 9. *Assume RH_F. Then Conjecture 11 implies Conjecture 8, and therefore also Conjectures 6 and 7.*

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