SIMPLE PROOF OF GALLAGHER'S SINGULAR SERIES SUM ESTIMATE

KEVIN FORD

P. X. Gallagher (Mathematika **23** (1976), 4–9) proved an estimate for the average of the singular series associated with r-tuples of linear forms, namely

$$\sum_{\substack{0\le h_1,\dots,h_r\le h\\ h_1,\dots,h_r \text{ distinct}}} \prod_p \left(1-\frac{\nu_p(\mathbf{h})}{p}\right)\left(1-\frac{1}{p}\right)^{-r} \sim h^r \tag{1}$$

for each fixed r, where $\nu_p(\mathbf{h})$ is the number of residue classes modulo p occupied by the numbers $h_1,\dots,h_r$. We give a simpler proof of this result below, with a worse error estimate than Gallagher obtained. The constants in all O-terms may depend on r.

Put $y=\frac12\log h$. We first note that $\nu_p(\mathbf{h})=r$ if $p\nmid H$, where $H=\prod_{i<j}|h_i-h_j|$. The number of prime factors of H is $O(\log H/\log\log H)=O(\log h/\log\log h)$. For any $h_1,\dots,h_r$, we therefore have

$$\begin{aligned}\prod_{p>y}\left(1-\frac{\nu_p(\mathbf{h})}{p}\right)\left(1-\frac{1}{p}\right)^{-r} &= \prod_{p\mid H,\, p>y}\left(1+O\left(\frac{1}{p}\right)\right)\prod_{p\nmid H,\, p>y}\left(1+O\left(\frac{1}{p^2}\right)\right)\\ &= 1+O\left(\frac{\log h}{y\log\log h}\right)=1+O\left(\frac{1}{\log\log h}\right).\end{aligned}$$

Thus, the left side of (1) is equal to AB, where

$$A=\left(1+O\left(\frac{1}{\log\log h}\right)\right)\prod_{p\le y}\left(1-\frac{1}{p}\right)^{-r},\quad B=\sum_{\substack{0\le h_1,\dots,h_r\le h\\ h_1,\dots,h_r \text{ distinct}}}\prod_{p\le y}\left(1-\frac{\nu_p(\mathbf{h})}{p}\right).$$

We have $B=O(h^{r-1})+B'$, where B' is the corresponding sum without the condition that $h_1,\dots,h_r$ are distinct. Let $P=\prod_{p\le y}p$ and note that $P=e^{y+o(y)}=h^{1/2+o(1)}$. The product in B is $1/P$ times the number of n, $0\le n<P$, satisfying $(\prod_i(n+h_i),P)=1$. Threrefore,

$$\begin{aligned}B' &= \sum_{0\le h_1,\dots,h_r\le h}\frac{1}{P}\sum_{n=0}^{P-1}\prod_{i=1}^{r}\sum_{d_i\mid(n+h_i,P)}\mu(d_i)\\ &= \frac{1}{P}\sum_{n=0}^{P-1}\sum_{d_1,\dots,d_r\mid P}\mu(d_1)\cdots\mu(d_r)\prod_{i=1}^{r}\left(\frac{h}{d_i}+O(1)\right)\\ &= h^r\sum_{d_1,\dots,d_r\mid P}\frac{\mu(d_1)\cdots\mu(d_r)}{d_1\cdots d_r}+O\left(h^{r-1}\sum_{d_1,\dots,d_{r-1}\mid P}\frac{1}{d_1\cdots d_r}\right)\\ &= h^r\prod_{p\le y}\left(1-\frac{1}{p}\right)^r+O(h^{r-1+o(1)}).\end{aligned}$$

Combined with the expression for A, this proves (1).

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