THE DISTRIBUTION OF INTEGERS WITH AT LEAST TWO DIVISORS IN A SHORT INTERVAL

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Abstract. We estimate the density of integers which have more than one divisor in an interval $(y,z]$ with $z \approx y + y/(\log y)^{\log 4 - 1}$. As a consequence, we determine the precise range of z such that most integers which have at least one divisor in $(y,z]$ have exactly one such divisor.

1. Introduction

Whereas, in usual cases, sieving by a set of primes may be fairly well controlled, through Buchstab's identity, sieving by a set of integers is a much more complicated task. However, some fairly precise results are known in the case where the set of integers is an interval. We refer to the recent work [1] of the first author for specific statements and references.

Define

$$\begin{aligned} \tau(n;y,z) &:= |\{d|n : y < d \leqslant z\}|, \\ H(x,y,z) &:= |\{n \leqslant x : \tau(n;y,z) \geqslant 1\}|, \\ H_r(x,y,z) &:= |\{n \leqslant x : \tau(n;y,z) = r\}|, \\ H_2^*(x,y,z) &:= |\{n \leqslant x : \tau(n;y,z) \geqslant 2\}| = \sum_{r\geqslant 2} H_r(x,y,z). \end{aligned}$$

Thus, the numbers $H_r(x,y,z)$ $(r \geqslant 1)$ describe the local laws of the function $\tau(n;y,z)$. When y and z are close, it is expected that, if an integer has at least a divisor in $(y,z]$, then it usually has exactly one, in other words

$$H(x,y,z) \sim H_1(x,y,z). \tag{1.1}$$

In this paper, we address the problem of determining the exact range of validity of such behavior. In other words, we search for a necessary and sufficient condition so that $H_2^*(x,y,z) = o\big(H(x,y,z)\big)$ as x and y tend to infinity. We show below that (1.1) holds if and only if

$$\lfloor y \rfloor + 1 \leqslant z < y + \frac{y}{(\log y)^{\log 4 - 1 + o(1)}} \qquad (y \to \infty).$$

As with the results in [1], the ratios $H(x,y,z)/x$ and $H_r(x,y,z)/x$ are weakly dependent on x when $x \geqslant y^2$. We take pains to prove results which are valid throughout the range $10 \leqslant y \leqslant \sqrt{x}$, since many interesting applications require bounds for $H(x,y,z)$ and $H_r(x,y,z)$ when $y \approx \sqrt{x}$; see e.g. §1 of [1] and Ch. 2 of [4] for some examples.

As shown in [6], for given y, the threshold for the behavior of the function $H(x,y,z)$ lies near the critical value

$$z = z_0(y) := y \exp\{(\log y)^{1-\log 4}\} \approx y + y/(\log y)^{\log 4 - 1}.$$

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We concentrate on the case $z_0(y) \leqslant z \leqslant \mathrm{e}y$. Define

$$\begin{aligned} &z = \mathrm{e}^{\eta} y, \quad \eta = (\log y)^{-\beta}, \quad \beta = \log 4 - 1 - \Xi/\sqrt{\log_2 y}, \quad \lambda = \frac{1+\beta}{\log 2}, \\ &Q(w) = \int_1^w \log t \,\mathrm{d}t = w\log w - w + 1. \end{aligned} \tag{1.2}$$

Here $\log_k$ denotes the kth iterate of the logarithm.

With the above notation, we have

$$\log(z/y) = \frac{\mathrm{e}^{\Xi\sqrt{\log_2 y}}}{(\log y)^{\log 4 - 1}}, \qquad \log(z/z_0(y)) = \frac{\mathrm{e}^{\Xi\sqrt{\log_2 y}} - 1}{(\log y)^{\log 4-1}},$$

so

$$0 \leqslant \Xi \leqslant (\log 4 - 1)\sqrt{\log_2 y}, \tag{1.3}$$

$$0 \leqslant \beta \leqslant \log 4 - 1, \tag{1.4}$$

$$\frac{1}{\log 2} \leqslant \lambda \leqslant 2. \tag{1.5}$$

From Theorem 1 of [1], we know that, uniformly in $10 \leqslant y \leqslant \sqrt{x}$, $z_0(y) \leqslant z \leqslant \mathrm{e}y$,

$$H(x,y,z) \asymp \frac{\beta x}{(\Xi+1)(\log y)^{Q(\lambda)}}. \tag{1.6}$$

By Theorems 5 and 6 of [1], for any $c > 0$ and uniformly in $y_0(r) \leqslant y \leqslant x^{1/2-c}$, $z_0(y) \leqslant z \leqslant \mathrm{e}y$ for a suitable constant $y_0(r)$, we have

$$\begin{aligned} &\frac{H_1(x,y,z)}{H(x,y,z)} \asymp_c 1, \\ \frac{\Xi+1}{\sqrt{\log_2 y}} \ll_{r,c} &\frac{H_r(x,y,z)}{H(x,y,z)} \leqslant 1 \quad (r \geqslant 2). \end{aligned} \tag{1.7}$$

When $0 \leqslant \Xi \leqslant o(\sqrt{\log_2 y})$ and $r \geqslant 2$, the upper and lower bounds above for $H_r(x,y,z)$ have different orders. We show in this paper that the lower bound represents the correct order of magnitude.

Theorem 1. *Uniformly in $10 \leqslant y \leqslant \sqrt{x}$, $z_0(y) \leqslant z \leqslant \mathrm{e}y$, we have*

$$\frac{H_2^*(x,y,z)}{H(x,y,z)} \ll \frac{\Xi+1}{\sqrt{\log_2 y}}$$

where $\Xi = \Xi(y,z)$ is defined as in (1.2) *and therefore satisfies* (1.3).

Corollary 2. *Let $r \geqslant 2$ and $c > 0$. There exists a constant $y_0(r,c)$ such that, uniformly for $y_0(r,c) \leqslant y \leqslant x^{1/2-c}$, $z_0(y) \leqslant z \leqslant \mathrm{e}y$, we have*

$$\frac{H_r(x,y,z)}{H(x,y,z)} \asymp_{r,c} \frac{\Xi+1}{\sqrt{\log_2 y}}.$$

Theorem 1 tells us that $H_2^*(x,y,z) = o\big(H(x,y,z)\big)$ whenever $z \geqslant z_0(y)$ and $\Xi = o\big(\sqrt{\log_2 y}\big)$. It is a simple matter to adapt the proofs given in [5] to show that this latter relation persists in the range $\lfloor y \rfloor + 1 \leqslant z \leqslant z_0(y)$. We thus obtain the following statement.

Corollary 3. *If $y \to \infty$, $y \leqslant \sqrt{x}$, and $\lfloor y \rfloor + 1 \leqslant z \leqslant y + y(\log y)^{1-\log 4+o(1)}$, we have*

$$H_1(x,y,z) \sim H(x,y,z).$$

Since we know from (1.7) that $H_2^*(x,y,z) \gg_\varepsilon H(x,y,z)$ when $\beta \leqslant \log 4 - 1 - \varepsilon$ for any fixed $\varepsilon > 0$ we have therefore completely answered the question raised at the beginning of this introduction concerning the exact validity range for the asymptotic formula (1.1). This may be viewed as a complement to a theorem of Hall (see [3], ch. 7; following a note mentioned by Hall in private correspondence, we slightly modify the statement) according to which

$$H(x,y,z) \sim F(-\Xi) \sum_{r \geqslant 1} r H_r(x,y,z) = F(-\Xi) \sum_{n \leqslant x} \tau(n;y,z) \tag{1.8}$$

in the range $\Xi = o(\log_2 y)^{1/6}$, $x > \exp\{\log z \log_2 z\}$ with

$$F(\xi) := \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\xi/\log 4} \mathrm{e}^{-u^2}\,\mathrm{d}u.$$

It is likely that (1.8) still holds in the range $(\log_2 y)^{1/6} \ll \Xi \leqslant o\big(\sqrt{\log_2 y}\big)$.

2. Auxiliary estimates

In the sequel, unless otherwise indicated, constants implied by Landau's O and Vinogradov's $\ll$ symbols are absolute and effective. Numerical values of reasonable size could easily be given if needed.

Let m be a positive integer. We denote by $P^-(m)$ the smallest, and by $P^+(m)$ the largest, prime factor of m, with the convention that $P^-(1) = \infty$, $P^+(1) = 1$. We write $\omega(m)$ for the number of distinct prime factors of m and $\Omega(m)$ for the number of prime power divisors of m. We further define

$$\omega(m;t,u) = \sum_{\substack{p^\nu \| m \\ t < p \leqslant u}} 1, \quad \Omega(m;t,u) = \sum_{\substack{p^\nu \| m \\ t < p \leqslant u}} \nu, \quad \overline{\Omega}(m;t) = \Omega(m;2,t), \quad \overline{\Omega}(m) = \Omega(m;2,m).$$

Here and in the sequel, the letter p denotes a prime number. Also, we let $\mathscr{P}(u,v)$ denote the set of integers all of whose prime factors are in $(u,v]$ and write $\mathscr{P}^*(u,v)$ for the set of squarefree members of $\mathscr{P}(u,v)$. By convention, $1 \in \mathscr{P}^*(u,v)$.

Lemma 2.1. *There is an absolute constant $C > 0$ so that for $\frac{3}{2} \leqslant u < v$, $v \geqslant \mathrm{e}^4$, $0 \leqslant \alpha \leqslant 1/\log v$, we have*

$$\sum_{\substack{m \in \mathscr{P}(u,v) \\ \omega(m) = k}} \frac{1}{m^{1-\alpha}} \leqslant \frac{(\log_2 v - \log_2 u + C)^k}{k!}.$$

Proof. For a prime $p \leqslant v$, we have $p^\alpha \leqslant 1 + 2\alpha \log p$, thus the sum in question is

$$\leqslant \frac{1}{k!}\bigg(\sum_{u < p \leqslant v} \frac{1}{p^{1-\alpha}} + \frac{1}{p^{2-2\alpha}} + \cdots\bigg)^k \leqslant \frac{\{\log_2 v - \log_2 u + O(1)\}^k}{k!}.$$

□

We note incidentally that a similar lower bound is available when u and v are not too close. See for instance Lemma III.13 of [2].

Lemma 2.2. *Uniformly for $u \geqslant 10$, $0 \leqslant k \leqslant 2.9 \log_2 u$, and $0 \leqslant \alpha \leqslant 1/(100 \log u)$, we have*

$$\sum_{\substack{P^+(m) \leqslant u \\ \overline{\Omega}(m) = k}} \frac{1}{m^{1-\alpha}} \ll \frac{(\log_2 u)^k}{k!}.$$

Proof. We follow the proof of Theorem 08 of [4]. Let w be a complex number with $|w| \leqslant \frac{29}{10}$. If p is prime and $3 \leqslant p \leqslant u$, then $|w/p^{1-\alpha}| \leqslant \frac{99}{100}$ and $p^\alpha \leqslant 1 + 2\alpha \log p$. Thus,

$$S(w) := \sum_{P^+(m)\leqslant u} \frac{w^{\overline{\Omega}(m)}}{m^{1-\alpha}} = \Big(1 - \frac{1}{2^{1-\alpha}}\Big)^{-1} \prod_{3\leqslant p\leqslant u} \Big(1 - \frac{w}{p^{1-\alpha}}\Big)^{-1} \ll \mathrm{e}^{(\Re w)\log_2 u}.$$

Put $r := k/\log_2 u$. By Cauchy's formula and Stirling's formula,

$$\sum_{\substack{P^+(m)\leqslant u\\ \overline{\Omega}(m)=k}} \frac{1}{m^{1-\alpha}} = \frac{1}{2\pi r^k}\int_{-\pi}^{\pi} \mathrm{e}^{-ik\vartheta} S(r\mathrm{e}^{i\vartheta})\, d\vartheta \ll \frac{(\log_2 u)^k}{k^k}\int_{-\pi}^{\pi} \mathrm{e}^{k\cos\vartheta}\, d\vartheta \ll \frac{(\log_2 u)^k}{k!}.$$

□

Lemma 2.3. *Suppose z is large, $0 \leqslant a + b \leqslant \frac{5}{2}\log_2 z$ and*

$$\exp\{(\log x)^{9/10}\} \leqslant w \leqslant z \leqslant x, \qquad xz^{-1/(10\log_2 z)} \leqslant Y \leqslant x.$$

The number of integers n with $x - Y < n \leqslant x$, $\overline{\Omega}(n; w) = a$ and $\omega(n; w, z) = \Omega(n; w, z) = b$, is

$$\ll \frac{Y}{\log z}\frac{\{\log_2 w\}^a}{a!}\frac{(b+1)\{\log_2 z - \log_2 w + C\}^b}{b!},$$

where C is a positive absolute constant.

Proof. There are $\ll x^{9/10}$ integers with $n \leqslant x^{9/10}$ or $2^j|n$ with $2^j \geqslant x^{1/10}$. For other n, write $n = rst$, where $P^+(r) \leqslant w$, $s \in \mathscr{P}^*(w, z)$ and $P^-(t) > z$. Here $\overline{\Omega}(r) = a$ and $\omega(s) = b$. We have either $t = 1$ or $t > z$. In the latter case $x/rs > z$, whence $Y/rs > \sqrt{z}$. We may therefore apply a standard sieve estimate to bound, for given r and s, the number of t by

$$\ll \frac{Y}{rs\log z}.$$

By Lemmas 2.1 and 2.2,

$$\sum_{r,s}\frac{1}{rs} \ll \frac{(\log_2 w)^a(\log_2 z - \log_2 w + C)^b}{a!b!}.$$

If $t = 1$, then we may assume $a + b \geqslant 1$. Set $p = P^+(n)$. If $b \geqslant 1$, then $p|s$ and we put $r_1 := r$ and $s_1 := s/p$. Otherwise, let $r_1 := r/p$ and $s_1 := s = 1$. Let $A := \overline{\Omega}(r_1)$ and $B := \omega(s_1)$, so that $A + B = a + b - 1$ in all circumstances. We have

$$p \geqslant x^{1/2\overline{\Omega}(n)} \geqslant x^{1/5\log_2 z} \geqslant (x/Y)^2.$$

Define the non-negative integer h by $z^{\mathrm{e}^{-h-1}} < p \leqslant z^{\mathrm{e}^{-h}}$. By the Brun-Titchmarsh theorem, we see that, for each given h, r_1 and s_1, the number of p is $\ll Y\mathrm{e}^h/(r_1s_1\log z)$. Set $\alpha := 0$ if $h = 0$ and $\alpha := \mathrm{e}^h/(100\log z)$ otherwise. For $h \geqslant 1$, we have $r_1s_1 > x^{3/4}z^{-1/\mathrm{e}} > \sqrt{z}$. Therefore, for $h \geqslant 0$,

$$\frac{1}{r_1s_1} \leqslant \frac{z^{-\alpha/2}}{(r_1s_1)^{1-\alpha}} \ll \frac{\mathrm{e}^{-\mathrm{e}^h/200}}{(r_1s_1)^{1-\alpha}}.$$

Now, Lemmas 2.1 and 2.2 imply that

$$\sum_{r_1,s_1}\frac{1}{(r_1s_1)^{1-\alpha}}\ll\frac{(\log_2 w)^A(\log_2 z-\log_2 w+C)^B}{A!B!}$$
$$\ll(b+1)\frac{(\log_2 w)^a(\log_2 z-\log_2 w+C)^b}{a!b!},$$

where we used the fact that $a\ll\log_2 w$. Summing over all h, we derive that the number of those integers $n>x^{9/10}$ satisfying the conditions of the statement is

$$\ll\frac{Y}{\log z}(b+1)\frac{(\log_2 w)^a(\log_2 z-\log_2 w+C)^b}{a!b!}.$$

Since $a!b!\leqslant(3\log_2 z)^{3\log_2 z}$, this last expression is $>x^{9/10}$. This completes the proof. □

Our final lemma is a special case of a theorem of Shiu (Theorem 03 of [4]).

Lemma 2.4. *Let f be a multiplicative function such that $0\leqslant f(n)\leqslant 1$ for all n. Then, for all x, Y with $1<\sqrt{x}\leqslant Y\leqslant x$, we have*

$$\sum_{x-Y<n\leqslant x}f(n)\ll\frac{Y}{\log x}\exp\Big\{\sum_{p\leqslant x}\frac{f(p)}{p}\Big\}.$$

3. Decomposition and outline of the proof

Throughout, ε will denote a very small positive constant. Note that Theorem 1 holds trivially for $\beta\leqslant\log 4-1-\varepsilon$ since we then have $1\ll\Xi/\log_2 y$ and of course $H_2^*(x,y,z)\leqslant H(x,y,z)$. We may henceforth assume that

$$\log 4-1-\varepsilon\leqslant\beta\leqslant\log 4-1. \tag{3.1}$$

Let

$$K:=\lfloor\lambda\log_2 z\rfloor,$$

so that $(2-\frac{3}{2}\varepsilon)\log_2 z\leqslant K\leqslant 2\log_2 z$. In light of (1.6), Theorem 1 reduces to

$$H_2^*(x,y,z)\ll\frac{x}{(\log y)^{Q(\lambda)}\sqrt{\log_2 y}}. \tag{3.2}$$

At this stage, we notice for further reference that, by Stirling's formula, for $k\leqslant K$ we have

$$\frac{\eta(2\log_2 z)^k}{k!(\log z)^2}\leqslant\frac{\eta(2\log_2 z)^K}{K!(\log z)^2}\asymp\frac{1}{(\log y)^{Q(\lambda)}\sqrt{\log_2 y}}. \tag{3.3}$$

Let $\mathcal{H}$ denote the set of integers $n\leqslant x$ with $\tau(n;y,z)\geqslant 2$. We count separately the integers $n\in\mathcal{H}$ lying in 6 classes. In these definitions, we write $k=\overline{\Omega}(n;z)$, $b=K-k$ and for brevity we put $z_h=z^{\mathrm{e}^{-h}}$. Let

$$K_0:=(2-3\varepsilon)\log_2 z$$

and define

$$\begin{aligned}
\mathcal{N}_0 &:= \{n \in \mathcal{H} : n \leqslant x/\log z \text{ or } \exists d > \log z : d^2|n\},\\
\mathcal{N}_1 &:= \{n \in \mathcal{H} \smallsetminus \mathcal{N}_0 : k \notin (K_0, K]\},\\
\mathcal{N}_2 &:= \bigcup_{1\leqslant h\leqslant 5\varepsilon \log_2 z} \mathcal{N}_{2,h},\\
\text{with } \mathcal{N}_{2,h} &:= \Big\{n \in \mathcal{H} \smallsetminus (\mathcal{N}_0 \cup \mathcal{N}_1) : \overline{\Omega}(n; z_h, z) \leqslant \tfrac{19}{10}h - \tfrac{1}{100}b\Big\}.
\end{aligned}$$

For integers $n \in \mathcal{N}_2$, we will only use the fact that $\tau(n; y, z) \geqslant 1$. Integers in other classes do not have too many small prime factors and it is sufficient to count pairs of divisors d_1, d_2 of n in $(y, z]$. For each such pair, write $v = (d_1, d_2)$, $d_1 = vf_1$, $d_2 = vf_2$, $n = f_1f_2vu$ and assume $f_1 < f_2$. Let

$$F_1 = \overline{\Omega}(f_1), \quad F_2 = \overline{\Omega}(f_2), \quad V = \overline{\Omega}(v), \quad U = \overline{\Omega}(u, z), \tag{3.4}$$

and

$$Z := \exp\{(\log z)^{1-4\varepsilon}\}. \tag{3.5}$$

For further reference, we note that if $n \notin \mathcal{N}_0$ and $h \leqslant 5\varepsilon \log_2 z$, then

$$\overline{\Omega}(n; z_h, z) = \omega(n; z_h, z).$$

Now we define $\mathcal{H}^* := \mathcal{H} \smallsetminus (\mathcal{N}_0 \cup \mathcal{N}_1 \cup \mathcal{N}_2)$ and

$$\begin{aligned}
\mathcal{N}_3 &:= \{n \in \mathcal{H}^* : \min(u, f_2) \leqslant Z\},\\
\mathcal{N}_4 &:= \{n \in \mathcal{H}^* : \min(u, f_2) > z^{1/10}\},\\
\mathcal{N}_5 &:= \{n \in \mathcal{H}^* : Z < \min(u, f_2) \leqslant z^{1/10}\}.
\end{aligned}$$

In the above decomposition, the main parts are $\mathcal{N}_2$ and $\mathcal{N}_5$. We expect $\mathcal{N}_2$ to be small since, conditionally on $\overline{\Omega}(n; z) = k$, the normal value of $\overline{\Omega}(n; z_h, z)$ is $hk/\log_2 z > \frac{19}{10}h$. It is more difficult to see that $\mathcal{N}_5$ is small too. This follows from the fact that we count integers in this set according to their number of factorizations in the form $n = uvf_1f_2$ with $y < vf_1 < vf_2 \leqslant z$. Suppose for instance that $f_1, f_2 \leqslant z_j$. For $\overline{\Omega}(n; z) = k$ and $\overline{\Omega}(n; z_j, z) = G$, then, ignoring the given information on the localization of vf_1 and vf_2 in $(y, z]$, there are $4^{k-G}2^G = 4^k2^{-G}$ such factorizations. Thus, larger G means fewer factorizations. On probabilistic grounds, larger G should also mean fewer factorizations when information on the localization of vf_1 and vf_2 is available.

We now briefly consider the cases of $\mathcal{N}_0$ and $\mathcal{N}_1$.

Trivially,

$$|\mathcal{N}_0| \leqslant \frac{x}{\log z} + \sum_{d>\log z} \frac{x}{d^2} \ll \frac{x}{\log z} \ll \frac{x}{(\log y)^{Q(\lambda)}\sqrt{\log_2 y}}, \tag{3.6}$$

since $Q(\lambda) \leqslant Q(2) = \log 4 - 1$ in the range under consideration.

By the argument on pages 40–41 of [4],

$$\sum_{\substack{n\leqslant x\\ \overline{\Omega}(n;z)>K}} 1 \ll \frac{x}{(\log y)^{Q(\lambda)}\sqrt{\log_2 y}}.$$

Setting $t := 1 - \frac{3}{2}\varepsilon$, Lemma 2.4 gives

$$\sum_{\substack{n\leqslant x\\ \tau(n;y,z)\geqslant 1\\ \overline{\Omega}(n;z)\leqslant K_0}} 1 \leqslant t^{-(2-3\varepsilon)\log_2 z} \sum_{\substack{dm\leqslant x\\ y<d\leqslant z}} t^{\overline{\Omega}(d)+\overline{\Omega}(m;z)} \ll x(\log z)^{2t-2-\beta-(2-3\varepsilon)\log t}$$

$$\ll x(\log y)^{-\beta-2\varepsilon^2} \ll x(\log y)^{-Q(\lambda)-\varepsilon^2/2}.$$

Therefore,

$$|\mathcal{N}_1| \ll \frac{x}{(\log y)^{Q(\lambda)}\sqrt{\log_2 y}}. \tag{3.7}$$

In the next four sections, we show that

$$|\mathcal{N}_j| \ll \frac{x}{(\log y)^{Q(\lambda)}\sqrt{\log_2 y}} \qquad (2\leqslant j\leqslant 5). \tag{3.8}$$

Together with (3.6) and (3.7), this will complete the proof of Theorem 1.

4. Estimation of $|\mathcal{N}_2|$

We plainly have $|\mathcal{N}_2| \leqslant \sum_h |\mathcal{N}_{2,h}|$. For $1\leqslant h\leqslant 5\varepsilon\log_2 z$, the numbers $n\in\mathcal{N}_{2,h}$ satisfy

$$\begin{cases} x/\log z < n \leqslant x,\\ k := \overline{\Omega}(n;z) = K-b, \quad 0\leqslant b\leqslant 3\varepsilon\log_2 z,\\ \overline{\Omega}(n;z_h,z) \leqslant \frac{19}{10}h - \frac{1}{100}b, \end{cases}$$

We note at the outset that $\mathcal{N}_{2,h}$ is empty unless $h\geqslant b/190$.

Write $n = du$ with $y<d\leqslant z$ and $u\leqslant x/y$. Let

$$\overline{\Omega}(d;z_h) = D_1, \quad \Omega(d;z_h,z) = D_2, \quad \overline{\Omega}(u;z_h) = U_1, \quad \Omega(u;z_h,z) = U_2,$$

so that $D_1+D_2\geqslant 1$, $D_2+U_2\leqslant \frac{19}{10}h - \frac{1}{100}b$ and $D_1+D_2+U_1+U_2 = k$.

Fix $k = K-b$, h, D_1, D_2, U_1 and U_2. By Lemma 2.3 (with $w=z_h$, $a=U_1$, $b=U_2$), the number of u is

$$\ll \frac{x}{y\log z}\,\frac{(\log_2 z-h)^{U_1}}{U_1!}(U_2+1)\frac{(h+C)^{U_2}}{U_2!}.$$

A second application of Lemma 2.3 yields that the number of d is

$$\ll \frac{\eta y}{\log z}\frac{(\log_2 z-h)^{D_1}}{D_1!}(D_2+1)\frac{(h+C)^{D_2}}{D_2!}.$$

Since $D_2+U_2<2h$, we have $(h+C)^{U_2+D_2}\leqslant \mathrm{e}^{2C}h^{U_2+D_2}$. Summing over D_1,D_2,U_1,U_2 with $G = D_2+U_2$ fixed and using the binomial theorem, we find that the number of n in question is

$$\ll \frac{\eta x}{(\log z)^2}(\log_2 z-h)^{k-G}h^G(G+1)^2 \sum_{\substack{U_1+D_1=k-G\\ D_2+U_2=G}} \frac{1}{U_1!D_1!D_2!U_2!} \ll \frac{\eta x 2^k}{(\log z)^2}A(h,G),$$

where

$$A(h,G) = (G+1)^2\frac{(\log_2 z-h)^{k-G}h^G}{(k-G)!G!}.$$

Since $G+1\leqslant G_h := \left\lfloor\frac{19}{10}h\right\rfloor$, we have

$$\frac{A(h,G+1)}{A(h,G)} \geqslant \frac{h(k-G)}{(G+1)(\log_2 z-h)} \geqslant \frac{k-10\varepsilon\log_2 z}{1.9(1-5\varepsilon)\log_2 z} > \tfrac{21}{20}$$

if ε is small enough. Next,

$$\begin{aligned} A(h,G_h) &\leqslant (G_h+1)^2\frac{(\log_2 z-h)^{k-G_h}(hk)^{G_h}}{k!(G_h/\mathrm{e})^{G_h}} \\ &\ll (h+1)^2\frac{(\log_2 z)^k}{k!}\big(\tfrac{20}{19}\mathrm{e}\big)^{19h/10}\mathrm{e}^{-h(k-G_h)/\log_2 z} \\ &\ll \frac{(\log_2 z)^k}{k!}\mathrm{e}^{-h/500}, \end{aligned}$$

since $(k-G_h)/\log_2 z > 2-13\varepsilon$ and $\frac{19}{10}\log(\frac{20}{19}\mathrm{e}) < 2-1/400$. Thus,

$$\sum_{b/190\leqslant h\leqslant 5\varepsilon\log_2 z}\ \sum_{0\leqslant G\leqslant G_h} A(h,G) \ll \sum_{b/190\leqslant h\leqslant 5\varepsilon\log_2 z} A(h,G_h) \ll \frac{(\log_2 z)^k}{k!}\mathrm{e}^{-b/95000}$$

and so

$$\sum_{\substack{n\in\mathcal{N}_2\\ \overline{\Omega}(n;z)=k}} 1 \ll \frac{\eta x(2\log_2 z)^k}{(\log z)^2 k!}\mathrm{e}^{-(K-k)/95000} \ll \frac{x\mathrm{e}^{-(K-k)/95000}}{(\log y)^{Q(\lambda)}\sqrt{\log_2 y}},$$

by (3.3). Summing over the range $K_0\leqslant k\leqslant K$ furnishes the required estimate (3.8) for $j=2$.

5. Estimation of $|\mathcal{N}_3|$

All integers $n=f_1f_2uv$ counted in $\mathcal{N}_3$ verify

$$\begin{cases} x/\log z < n\leqslant x, \\ \overline{\Omega}(n;z)\leqslant K, \\ y<vf_1<vf_2\leqslant z, \quad \min(u,f_2)\leqslant Z, \end{cases}$$

where Z is defined in (3.5). This is all we shall use in bounding $|\mathcal{N}_3|$.

Let $\mathcal{N}_{3,1}$ be the subset corresponding to the condition $f_2\leqslant Z$ and let $\mathcal{N}_{3,2}$ comprise those $n\in\mathcal{N}_3$ such that $u\leqslant Z$.

If $f_2\leqslant Z$, then $v>z^{1/2}$ and $u>x/\{vZ^2\log z\}>x^{1/3}$. For $\frac12\leqslant t\leqslant 1$ we have

$$\begin{aligned} |\mathcal{N}_{3,1}| &\leqslant \sum_{f_1,f_2,v,u} t^{\overline{\Omega}(f_1f_2uv;z)-K} \\ &= t^{-K}\sum_{f_1\leqslant Z} t^{\overline{\Omega}(f_1)} \sum_{f_1<f_2\leqslant \mathrm{e}^{\eta}f_1} t^{\overline{\Omega}(f_2)} \sum_{y/f_1<v\leqslant z/f_1} t^{\overline{\Omega}(v)} \sum_{u\leqslant x/f_1f_2v} t^{\overline{\Omega}(u;z)}. \end{aligned}$$

Apply Lemma 2.4 to the three innermost sums. The u-sum is

$$\ll \frac{x}{f_1f_2v}(\log z)^{t-1} \leqslant \frac{x}{f_1y}(\log z)^{t-1}.$$

and the v-sum is

$$\ll \frac{\eta y}{f_1}(\log z)^{t-1}.$$

The f_2-sum is $\ll \eta f_1(\log f_1)^{t-1}$ if $f_1 > \eta^{-3}$ and otherwise is $\ll \eta f_1$ trivially (note that $\eta f_1 \gg 1$ follows from the fact that $(f_1+1)/f_1 \leqslant f_2/f_1 \leqslant \mathrm{e}^{\eta}$). Next

$$\sum_{f_1\leqslant \eta^{-3}} \frac{1}{f_1} + \sum_{2\leqslant f_1\leqslant Z} \frac{t^{\overline{\Omega}(f_1)}}{f_1}(\log f_1)^{t-1} \ll \log_2 z + (\log_2 z) \max_{j\leqslant \log_2 Z} \mathrm{e}^{j(t-1)} \sum_{f_1\leqslant \exp\{\mathrm{e}^j\}} \frac{t^{\overline{\Omega}(f_1)}}{f_1}$$
$$\ll (\log_2 z)(\log Z)^{2t-1}.$$

Thus,

$$|\mathcal{N}_{3,1}| \ll x(\log_2 x)(\log x)^E$$

with $E = -2\beta - \lambda\log t + 2t - 2 + (2t-1)(1-4\varepsilon)$. We select optimally $t := \frac{1}{4}\lambda/(1-2\varepsilon)$, and check that $t \geqslant \frac{1}{2}$ since $\lambda \geqslant 2 - \varepsilon/\log 2$. Then

$$\begin{aligned} E &= -Q(\lambda) + \lambda\log(1-2\varepsilon) + 4\varepsilon \leqslant -Q(\lambda) + (2-\varepsilon/\log 2)(-2\varepsilon - 2\varepsilon^2) + 4\varepsilon \\ &< -Q(\lambda) - \varepsilon^2. \end{aligned}$$

Next, we consider the case when $u \leqslant Z$. We observe that this implies

$$\tfrac{1}{4}vz^2 \leqslant vx \leqslant vn\log z = uf_1vf_2v\log z \leqslant Zz^2\log z$$

hence $v \leqslant 4Z\log z \leqslant Z^2$, and therefore

$$\min(f_1, f_2) > z^{1/2}.$$

Also, $z > x^{1/3}$ since $x/\log z < n = uvf_1f_2 \leqslant Zz^2$. Thus, for $\frac{1}{2} \leqslant t \leqslant 1$, we have

$$\begin{aligned} |\mathcal{N}_{3,2}| &\leqslant \sum_{f_1,f_2,v,u} t^{\overline{\Omega}(f_1f_2uv;z)-K} \\ &= t^{-K}\sum_{v\leqslant Z^2} t^{\overline{\Omega}(v)} \sum_{u\leqslant xv/y^2} t^{\overline{\Omega}(u)} \sum_{y/v<f_1\leqslant z/v} t^{\overline{\Omega}(f_1)} \sum_{y/v<f_2\leqslant z/v} t^{\overline{\Omega}(f_2)}. \end{aligned}$$

The sums upon f_1 and f_2 are each

$$\ll \frac{\eta y}{v}(\log z)^{t-1}$$

and the u-sum is

$$\ll \frac{xv}{y^2}(\log 2xv/y^2)^{t-1} \leqslant \frac{xv}{y^2}(\log 2v)^{t-1}.$$

Thus, selecting the same value $t := \frac{1}{4}\lambda/(1-2\varepsilon)$, we obtain

$$\begin{aligned} |N_{3,2}| &\ll t^{-K}x\eta^2(\log z)^{2t-1}\sum_{v\leqslant Z^2}\frac{t^{\overline{\Omega}(v)}(\log 2v)^{t-1}}{v} \\ &\ll x(\log_2 z)(\log z)^E \leqslant x(\log_2 z)(\log z)^{-Q(\lambda)-\varepsilon^2}. \end{aligned}$$

This completes the proof of (3.8) with $j=3$.

6. Estimation of $|\mathcal{N}_4|$

We now consider those integers $n=f_1f_2uv$ such that

$$\begin{cases} x/\log z<n\leqslant x,\\ k:=\overline{\Omega}(n;z)=K-b,\quad 0\leqslant b\leqslant 3\varepsilon\log_2 z,\\ y<vf_1<vf_2\leqslant z,\quad \min(u,f_2)>z^{1/10}.\end{cases}$$

With the notation (3.4), fix k, F_1, F_2, U and V. Here u,f_1 and f_2 are all $>\frac12 z^{1/10}$. By Lemma 2.3 (with $w=z$), for each triple f_1,f_2,v the number of u is

$$\ll \frac{x}{f_1f_2v\log z}\,\frac{(\log_2 z)^U}{U!}\cdot$$

Using Lemma 2.3 two more times, we obtain, for each v,

$$\sum_{y/v<f_1\leqslant z/v}\frac{1}{f_1}\sum_{y/v<f_2\leqslant z/v}\frac{1}{f_2}\ll\frac{\eta^2}{(\log z)^2}\,\frac{(\log_2 z)^{F_1+F_2}}{F_1!F_2!}\cdot$$

Now, Lemma 2.2 gives

$$\sum_v\frac1v\ll\frac{(\log_2 z)^V}{V!}\cdot$$

Gathering these estimates and using (3.3) yields

$$\begin{aligned}|\mathcal{N}_4|&\ll\frac{x\eta^2}{(\log z)^3}\sum_{(2-3\varepsilon)\log_2 z\leqslant k\leqslant K}\ \sum_{F_1+F_2+U+V=k}\frac{(\log_2 z)^k}{F_1!F_2!U!V!}\\ &=\frac{x\eta^2}{(\log z)^3}\sum_{(2-3\varepsilon)\log_2 z\leqslant k\leqslant K}\frac{(2\log_2 z)^k}{k!}2^k\\ &\ll\frac{x}{(\log y)^{Q(\lambda)}\sqrt{\log_2 y}}\,\frac{2^K\eta}{\log z}\ll\frac{x}{(\log y)^{Q(\lambda)}\sqrt{\log_2 y}}\cdot\end{aligned}$$

Thus (3.8) holds for $j=4$.

7. Estimation of $|\mathcal{N}_5|$

It is plainly sufficient to bound the number of those $n=f_1f_2uv$ satisfying the following conditions

$$\begin{cases} x/\log z<n\leqslant x,\\ k:=\overline{\Omega}(n;z)=K-b,\quad 0\leqslant b\leqslant 3\varepsilon\log_2 z,\\ \overline{\Omega}(n;z_h,z)>\frac{19}{10}h-\frac{1}{100}b\quad (1\leqslant h\leqslant 5\varepsilon\log_2 z)\\ y<vf_1<vf_2\leqslant z,\quad Z<\min(u,f_2)\leqslant z^{1/10}.\end{cases}$$

Define j by $z_{j+2}<\min(u,f_2)\leqslant z_{j+1}$. We have $1\leqslant j\leqslant 5\varepsilon\log_2 z$. Let $\mathcal{N}_{5,1}$ be the set of those n satisfying the above conditions with $u\leqslant z_{j+1}$ and let $\mathcal{N}_{5,2}$ be the complementary set, for which $f_2\leqslant z_{j+1}$.

If $u\leqslant z_{j+1}$, then $v\leqslant(z^2u\log z)/x\leqslant 4u\log z\leqslant z_j$ and $f_2>f_1>z^{1/2}$. Recall notation (3.4) and write

$$F_{11}:=\overline{\Omega}(f_1;z_j),\quad F_{12}:=\Omega(f_1;z_j,z),\quad F_{21}:=\overline{\Omega}(f_2;z_j),\quad F_{22}:=\Omega(f_2;z_j,z),$$

so that the initial condition upon $\overline{\Omega}(n; z_h, z)$ with $h = j$ may be rewritten as

$$F_{12} + F_{22} \geqslant G_j := \max(0, \lfloor \tfrac{19}{10} j - b/100 \rfloor).$$

We count those n in a dyadic interval $(X, 2X]$, where $x/(2\log z) \leqslant X \leqslant x$. Fix k, j, X, U, V, F_{rs} and apply Lemma 2.3 to sums over u, f_1, f_2. The number of n is question is

$$\begin{aligned} &\leqslant \sum_{v \leqslant z_j} \sum_{vX/z^2 \leqslant u \leqslant 2vX/y^2} \sum_{y/v < f_1 \leqslant z/v} \sum_{y/v < f_2 \leqslant z/v} 1 \\ &\ll \frac{\eta^2 X \mathrm{e}^j}{(\log z)^3} \frac{(\log_2 z - j)^{U + F_{11} + F_{21}}}{U! F_{11}! F_{21}!} (F_{12}+1)(F_{22}+1) \frac{(j+C)^{F_{12}+F_{22}}}{F_{12}! F_{22}!} \sum_{v \leqslant z_j} \frac{1}{v}. \end{aligned}$$

Bounding the v-sum by Lemma 2.2, and summing over X, U, V, F_{rs} with $F_{12} + F_{22} = G$ yields

$$|\mathcal{N}_{5,1}| \ll \frac{\eta^2 x}{(\log z)^3} \sum_{(2-3\varepsilon)\log_2 z \leqslant k \leqslant K} 4^k \sum_{1 \leqslant j \leqslant 5\varepsilon \log_2 z} \sum_{G_j \leqslant G \leqslant k} M(j, G),$$

where

$$M(j, G) := \mathrm{e}^j (G+1)^2 \frac{(\log_2 z - j)^{k-G} (j+C)^G}{2^G (k-G)! G!}.$$

Let $j_b = \lfloor \frac{1}{2} b + 100C + 100 \rfloor$. If $j \leqslant j_b$, then $j + C \leqslant \frac{99}{100}(j + C_b)$ with $C_b := 3C + 2 + \frac{b}{100}$ and, introducing $R := \max_{G \geqslant 0} \{(G+1)^2 (\frac{99}{100})^G\}$, we have

$$\begin{aligned} \sum_{1 \leqslant j \leqslant j_b} \sum_{G_j \leqslant G \leqslant k} M(j, G) &\leqslant R \sum_{1 \leqslant j \leqslant j_b} \mathrm{e}^j \sum_{0 \leqslant G \leqslant k} \frac{(\log_2 z - j)^{k-G} (j + C_b)^G}{2^G G! (k-G)!} \\ &\ll \frac{1}{k!} \sum_{1 \leqslant j \leqslant j_b} \mathrm{e}^j \left(\log_2 z - \tfrac{1}{2} j + \tfrac{1}{2} C_b\right)^k \\ &\ll \frac{(\log_2 z)^k}{k!} \sum_{1 \leqslant j \leqslant j_b} \mathrm{e}^{j + (b/200 - j/2)k/\log_2 z} \\ &\ll \frac{(\log_2 z)^k}{k!} \mathrm{e}^{b/100 + 2\varepsilon j_b} \ll \frac{(\log_2 z)^k}{k!} \mathrm{e}^{b/50}. \end{aligned}$$

When $j > j_b$, then

$$G_j \geqslant \tfrac{9}{5}(j + C) + \tfrac{1}{10}(j_b + C + 1) - \tfrac{1}{100} b - 1 \geqslant \tfrac{9}{5}(j + C) + 9 \geqslant 189.$$

Thus, for $G \geqslant G_j$ we have

$$\frac{M(j, G+1)}{M(j, G)} = \left(\frac{G+2}{G+1}\right)^2 \frac{j + C}{2(G+1)} \frac{k - G}{\log_2 z - j} \leqslant \tfrac{4}{7}.$$

Therefore,

$$\begin{aligned} \sum_{G_j \leqslant G \leqslant k} M(j, G) &\ll M(j, G_j) \ll \frac{j^2 \mathrm{e}^j}{k!} \frac{(\log_2 z - j)^{k - G_j} (jk)^{G_j}}{2^{G_j} G_j!} \\ &\leqslant \frac{j^2 \mathrm{e}^j (\log_2 z)^k}{k!} \mathrm{e}^{-j(k - G_j)/\log_2 z} \left(\frac{\mathrm{e} jk}{2 G_j \log_2 z}\right)^{G_j} \ll \frac{(\log_2 z)^k}{k!} \mathrm{e}^{-j/5}, \end{aligned}$$

since $k-G_j \geqslant (2-10\varepsilon)\log_2 z$, $\mathrm{e}jk/(2G_j\log_2 z) \leqslant \frac{5}{9}\mathrm{e}$, and $-1+\frac{19}{10}\log(\frac{5}{9}\mathrm{e}) < -\frac{1}{5}$. We conclude that

$$\sum_{1\leqslant j\leqslant 5\varepsilon\log_2 z}\ \sum_{G_j\leqslant G\leqslant k} M(j,G) \ll \frac{(\log_2 z)^k}{k!}\mathrm{e}^{b/50} \tag{7.1}$$

and hence, by (3.3),

$$|\mathcal{N}_{5,1}| \ll \frac{\eta^2 x}{(\log z)^3}\sum_{k\leqslant K}\frac{(2\log_2 z)^k}{k!}2^{K-b/2} \ll \frac{\eta^2 2^K x}{(\log z)^3}\frac{(2\log_2 z)^K}{K!} \ll \frac{x}{(\log y)^{Q(\lambda)}\sqrt{\log_2 y}}.$$

Now assume $f_2 \leqslant z_{j+1}$. Then $\min(u,v) > \sqrt{z}$. Fix F_1, F_2 and

$$\overline{\Omega}(v;z_j) = V_1, \quad \Omega(v;z_j,z) = V_2, \quad \overline{\Omega}(u;z_j) = U_1, \quad \Omega(u;z_j,z) = U_2.$$

By Lemma 2.3, given f_1, f_2 and v, the number of u is

$$\ll \frac{x}{f_1 f_2 v\log z}\frac{(\log_2 z - j)^{U_1}(U_2+1)(j+C)^{U_2}}{U_1!U_2!}.$$

Applying Lemma 2.3 again, for each f_1 we have

$$\sum_{\substack{f_1<f_2\leqslant \mathrm{e}^\eta f_1\\ y/f_1<v\leqslant z/f_1}} \frac{1}{f_2 v} \ll \frac{\eta^2\mathrm{e}^j}{(\log z)^2}\frac{(V_2+1)(\log_2 z - j)^{V_1+F_2}(j+C)^{V_2}}{V_1!V_2!F_2!}.$$

By Lemma 2.2,

$$\sum_{f_1\leqslant z_j}\frac{1}{f_1} \ll \frac{(\log_2 z - j)^{F_1}}{F_1!}.$$

Combine these estimates, and sum over $F_1, F_2, U_1, U_2, V_1, V_2$ with $V_2+U_2 = G$. As in the estimation of $|\mathcal{N}_{5,1}|$, sum over k, j, G using (3.3) and (7.1). We obtain

$$\begin{aligned}|\mathcal{N}_{5,2}| &\ll \frac{\eta^2 x}{(\log z)^3}\sum_{(2-3\varepsilon)\log_2 z\leqslant k\leqslant K} 4^k \sum_{1\leqslant j\leqslant 5\varepsilon\log_2 z}\ \sum_{G_j<G\leqslant k} M(j,G)\\ &\ll \frac{x}{(\log y)^{Q(\lambda)}\sqrt{\log_2 y}}.\end{aligned}$$

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