AN EXPLICIT SIEVE BOUND AND SMALL VALUES OF $\sigma(\phi(m))$

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Abstract. We prove an explicit sieve upper bound based on the large sieve of Montgomery and Vaughan [MV], and apply it to show that $\sigma(\phi(m)) \geqslant m/39.4$ for all positive integers m.

1. Introduction

In 1973, Montgomery and Vaughan [MV] proved a weighted version of the large sieve inequality which allowed them to prove a very simple version of the Brun-Titchmarsh inequality, namely

$$\pi(x+y;q,a) - \pi(x;q,a) < \frac{2y}{\phi(q)\log(y/q)},$$

where $\pi(x;q,a)$ is the number of primes $p \leqslant x$, $\equiv a \pmod q$. Using the same weighted large sieve, we prove a general explicit upper sieve bound for shifted primes (numbers $p-1$ for primes p).

Theorem 1. *Let S be a set of primes containing 2. Let $U(x) = U(x;S)$ be the number of primes $p \leqslant x$ with $p-1$ composed only of primes in S. Let*

$$H(x) = \prod_{p\leqslant x, p\in S} \left(1 - \frac{1}{p}\right).$$

Then, for $x > 1$,

$$U(x) < \frac{x}{I(x)(1+1/\log x)}, \qquad I(x) = \int_1^{\sqrt{x}} \frac{\log t}{t} H(t)\,dt.$$

In §3 we apply Theorem 1 to a problem of Mąkowski and Schinzel [MS], who asked if

$$\frac{\sigma(\phi(m))}{m} \geqslant \frac{1}{2} \tag{1.1}$$

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for all integers m. Here $\sigma(m)$ is the sum of the positive divisors of m and ϕ is Euler's totient function. Equality holds when $m = 2$. Moreover, Mąkowski and Schinzel [MS] show that

$$\liminf_{m\to\infty} \frac{\sigma(\phi(m))}{m} \leqslant \frac{1}{2} + \frac{1}{2^{34}-4}.$$

In 1989, Pomerance [P] showed that $\inf \frac{\sigma(\phi(m))}{m} > 0$. Here we prove a specific lower bound.

Theorem 2. *For all natural numbers m,*

$$\frac{\sigma(\phi(m))}{m} \geqslant \frac{1}{39.4}.$$

2. Explicit Sieve Bounds

The object of this section is to prove Theorem 1, the main tool being the weitghed large sieve inequality ([MV], Corollary 1). In what follows, $\mu(n)$ will denote the Möbius function, $s(m)$ will denote the product of the distinct primes dividing m, $\pi(x)$ is the number of primes $\leqslant x$, and $\gamma = 0.57721566\ldots$ denotes the Euler-Mascheroni constant. For primes p, $p^e \| n$ means $p^e | n$ and $p^{e+1} \nmid n$.

Lemma 2.1. *We have*

$$\sum_{s(n)\leqslant x} \frac{1}{n} = \sum_{n\leqslant x} \frac{\mu^2(n)}{\phi(n)} = \log x + c(x),$$

where $c(x) \geqslant 1$ for $x \geqslant 6$ and $c(x) \geqslant 0.3$ for $x \geqslant 1$.

Proof. For $x < 65$ this follows by a direct computation, where we use the fact that $c(x)$ is decreasing on each interval $[k-1,k)$. For larger x we use the inequality

$$c(x) \geqslant 1.10689 - \frac{6.927}{x}, \tag{2.1}$$

which we now prove. Let $T(x) = \sum_{s(m)\leqslant x} \frac{1}{m}$. Let $S(x)$ denote the sum of the reciprocals of those m for which

$$\prod_{p^e\|m, p\leqslant 3} p \prod_{p^e\|m, p>3} p^e \leqslant x.$$

Each such m may be written uniquely as $m = ab$ with $s(a)|6$, $(b,6) = 1$ and $b \leqslant x/s(a)$. Thus

$$T(x) \geqslant S(x) = \sum_{d|6} \frac{1}{\phi(d)} \sum_{\substack{b\leqslant x/d \\ (b,6)=1}} \frac{1}{b} = \sum_{d|6} \frac{1}{\phi(d)} \sum_{e|6} \frac{\mu(e)}{e} \sum_{n\leqslant x/(de)} \frac{1}{n}.$$

We will now make use of the fact that

$$\sum_{n\leqslant x}\frac{1}{n}=\log x+\gamma+E(x),\qquad |E(x)|\leqslant\frac{\gamma}{x}\quad(x>0),$$

which is an easy application of partial summation. Then

$$\begin{aligned}S(x)&=\sum_{d|6,e|6}\frac{\mu(e)}{e\phi(d)}\left(\log\left(\frac{x}{de}\right)+\gamma+E\left(\frac{x}{de}\right)\right)\\&=\log x+\gamma+\frac{\log 2}{2}+\frac{\log 3}{6}+\sum_{d|6,e|6}E\left(\frac{x}{de}\right)\frac{\mu(e)}{e\phi(d)}\\&\geqslant\log x+1.10689-\frac{\gamma}{x}\sum_{D|36}D\left|\sum_{\substack{d|6,e|6\\D=ed}}\frac{\mu(e)}{e\phi(d)}\right|\\&=\log x+1.10689-\frac{12\gamma}{x}.\end{aligned}$$

This immediately gives (2.1). □

We next require some explicit estimates of functions involving primes from the paper of Rosser and Schoenfeld [RS].

Lemma 2.2 [RS, Theorem 7]. *For all $x>1$,*

$$\frac{e^{-\gamma}}{\log x}\left(1+\frac{1}{\log^2 x}\right)^{-1}<\prod_{p\leqslant x}\left(1-\frac{1}{p}\right)<\frac{e^{-\gamma}}{\log x}\left(1+\frac{1}{\log^2 x}\right).$$

Lemma 2.3 [RS, Theorem 1, Corollary 1]. *For all $x>17$,*

$$\frac{x}{\log x}\leqslant\pi(x)\leqslant\frac{x}{\log x}\left(1+\frac{3}{2\log x}\right).$$

Proof of Theorem 1. For $x\geqslant 2$ and all S we have the upper bound

$$I(x)\leqslant\int_1^2\frac{\log t}{t}\,dt+\frac{1}{2}\int_2^{\sqrt{x}}\frac{\log t}{t}\,dt\leqslant\frac{\log^2 x+1.922}{16}. \tag{2.2}$$

First, $U(x)=0$ when $1<x<2$, so the theorem is trivial. When $2\leqslant x\leqslant e^{13}$, Lemma 2.3 and (2.2) give

$$\frac{x}{I(x)(1+1/\log x)}\geqslant\frac{16x}{(1+1/\log x)(1.922+\log^2 x)}>\pi(x)\geqslant U(x),$$

so the theorem follows in this case.

Next, suppose that $x > e^{13}$. As in [MV], let $Q = \sqrt{2x/3}$, so that $Q > 543$. Then $U(x) \leqslant \pi(Q) + V(x)$, where $V(x)$ is the number of integers $n \leqslant x$ such that $n \not\equiv 0 \pmod q$ for each prime $q \leqslant Q$, and $n \not\equiv 1 \pmod q$ for primes $q \leqslant Q$, $q \notin S$. By the weighted large sieve (Corollary 1 of [MV]), we have

$$U(x) \leqslant \pi(Q) + \frac{x}{L}, \tag{2.3}$$

where

$$L = \sum_{q \leqslant Q} \left(1 + \frac{3qQ}{2x}\right)^{-1} \mu^2(q) \prod_{p|q} \frac{\omega(p)}{p - \omega(p)}, \quad \omega(p) = \begin{cases} 1 & p \in S \\ 2 & p \notin S \end{cases}.$$

Let $\Omega(n)$ be the number of prime factors of n counted with multiplicity and let $\tau(n)$ be the number of positive divisors of n. Since $3qQ/(2x) = q/Q$,

$$\begin{aligned} L &= \sum_{q \leqslant Q} \frac{\mu^2(q)}{1 + q/Q} \prod_{\substack{p|q \\ p \in S}} \left(\frac{1}{p} + \frac{1}{p^2} + \cdots\right) \prod_{\substack{p|q \\ p \notin S}} \left(\frac{2}{p} + \frac{4}{p^2} + \cdots\right) \\ &= \sum_{s(d_1 d_2) \leqslant Q} \frac{1}{1 + s(d_1 d_2)/Q} \frac{2^{\Omega(d_2)}}{d_1 d_2} \\ &\geqslant \sum_{s(d_1 d_2) \leqslant Q} \frac{1}{1 + s(d_1 d_2)/Q} \frac{\tau(d_2)}{d_1 d_2} \\ &= \sum_{s(d_1 d_3 d_4) \leqslant Q} \frac{1}{1 + s(d_1 d_3 d_4)/Q} \frac{1}{d_1 d_3 d_4}. \end{aligned}$$

Here d_1 is composed only of primes in S, and $d_2 = d_3 d_4$ is composed only of primes not in S. Let $d_5 = d_1 d_3$, so that there are no restrictions on the prime factors of d_5. Since $s(d_4 d_5) \leqslant s(d_4) s(d_5)$, we have

$$L \geqslant \sum_{s(d_4) s(d_5) \leqslant Q} \frac{1}{1 + s(d_4) s(d_5)/Q} \frac{1}{d_4 d_5}. \tag{2.4}$$

Let

$$g(x) = \sum_{s(d) \leqslant x} \frac{1}{d} = \sum_{n \leqslant x} \frac{\mu^2(n)}{\phi(n)} = \log x + c(x), \qquad f(x) = \sum_{s(d_4) \leqslant x} \frac{1}{d_4}.$$

Since every $d \leqslant x$ can be written uniquely in the form $d = d_1 d_4$,

$$g(x) \leqslant \sum_{s(d_1) \leqslant x} \frac{1}{d_1} \sum_{s(d_4) \leqslant x} \frac{1}{d_4} = H(x)^{-1} f(x). \tag{2.5}$$

We next show that

$$r(y) := \sum_{s(d) \leqslant y} (1 + s(d)/y)^{-1} d^{-1} = \sum_{n \leqslant y} (1 + n/y)^{-1} \frac{\mu^2(n)}{\phi(n)} > \log y \quad (y \geqslant 2). \tag{2.6}$$

By partial summation

$$\begin{aligned} r(y) &= \int_{1^-}^{y} \frac{dg(t)}{1+t/y} \\ &= \int_1^y \frac{dt}{t(1+t/y)} + \frac{c(t)}{1+t/y}\Big|_{1^-}^{y} + y\int_1^y \frac{c(t)}{(y+t)^2}\,dt \\ &= \log\left(\frac{y+1}{2}\right) + \frac{c(y)}{2} + y\int_1^y \frac{c(t)}{(y+t)^2}\,dt. \end{aligned}$$

By Lemma 2.1, when $y \geqslant 14$, we have

$$\begin{aligned} r(y) - \log y &\geqslant -\log 2 + \frac{1}{2} + y\int_6^y \frac{1\,dt}{(y+t)^2} \\ &= -\log 2 + \frac{y}{y+6} \geqslant 0. \end{aligned}$$

For $2 \leqslant y < 14$, we use the fact that $r(y) - \log y$ is decreasing on each interval $[k, k+1)$ for integral k, then check that $r(k) - \log(k+1) \geqslant 0$ for $2 \leqslant k \leqslant 13$. This proves (2.6). Writing $q_4 = s(d_4)$, by (2.4), (2.5) and (2.6), we have

$$\begin{aligned} L &\geqslant \sum_{s(d_4)\leqslant Q/2} \frac{\log(Q/s(d_4))}{d_4} \\ &= \sum_{q_4\leqslant Q/2} \frac{\mu^2(q_4)}{\phi(q_4)} \log(Q/q_4) \\ &= \int_{1^-}^{Q/2} \log(Q/t)df(t) \\ &= f(Q/2)\log 2 + \int_1^{Q/2} \frac{f(t)}{t}\,dt \\ &\geqslant f(Q/2)\log 2 + \int_1^{Q/2} \frac{H(t)g(t)}{t}\,dt \\ &= f(Q/2)\log 2 + \int_1^{Q/2} \frac{H(t)\log t}{t}\,dt + \int_1^{Q/2} \frac{H(t)c(t)}{t}\,dt \\ &\geqslant \int_1^{Q/2} \frac{H(t)\log t}{t}\,dt + H(Q/2)\left(g(Q/2)\log 2 + \int_1^{Q/2} \frac{c(t)}{t}\,dt\right). \end{aligned}$$

Since $Q \geqslant 20$, Lemma 2.1 and a short computation give

$$\int_1^{Q/2} \frac{c(t)}{t}\,dt \geqslant \int_1^{10} \frac{c(t)}{t}\,dt + \int_{10}^{Q/2} \frac{dt}{t} > \log Q - 0.645.$$

Also,

$$\int_{Q/2}^{\sqrt{x}} \frac{H(t)\log t}{t}\,dt \leqslant \frac{H(Q/2)}{2}\left(\log^2\sqrt{x} - \log^2(Q/2)\right) \leqslant \left(\frac{1}{2}\log 6\right) H(Q/2)\log Q.$$

Therefore, since $g(Q/2) \geqslant \log(Q/2)$ and since $Q \geqslant 543$,

$$\begin{aligned} L - I(x) &\geqslant H(Q/2)\left\{(1+\log 2 - \tfrac12 \log 6)\log Q - 0.645 - \log^2 2\right\} \\ &\geqslant 0.618 H(Q)\log Q, \end{aligned}$$

whence by (2.3),

$$L(x) - I(x) \geqslant 0.618 H(Q) \log Q. \tag{2.7}$$

By Lemma 2.2, when $1 \leqslant t \leqslant u$,

$$H(t) = H(u) \prod_{\substack{p\in S \\ t<p\leqslant u}} \left(1-\frac1p\right)^{-1} \leqslant H(u)\frac{\log u}{\log t}\left(1+\frac{1}{\log^2 u}\right)\left(1+\frac{1}{\log^2 t}\right). \tag{2.8}$$

Taking $u = \sqrt{x}$ in (2.8) gives

$$\begin{aligned} I(x) &\leqslant H(\sqrt{x}) \int_e^{\sqrt{x}} \frac{\log\sqrt{x}}{t}\left(1+\frac{1}{\log^2\sqrt{x}}\right)\left(1+\frac{1}{\log^2 t}\right)dt + \int_1^e \frac{H(t)\log t}{t}\,dt \\ &= H(\sqrt{x})\log\sqrt{x}\left(1+\frac{1}{\log^2\sqrt{x}}\right)\left(\log\sqrt{x} - \frac{1}{\log\sqrt{x}}\right) + \frac{1+\log^2 2}{4} \\ &\leqslant H(\sqrt{x})\log^2\sqrt{x} + 0.3702. \end{aligned}$$

Since $\sqrt{x} \geqslant 665$, Lemma 2.2 gives $H(\sqrt{x})\log\sqrt{x} \geqslant 0.5484$, and thus

$$I(x) \leqslant 1.104 H(\sqrt{x})\log^2\sqrt{x} \leqslant 1.14 H(Q)\log Q \log\sqrt{x},$$

whence, by (2.7),

$$L \geqslant I(x)\left(1 + 1.08/\log x\right).$$

By (2.1), $I(x) \leqslant 0.0633 \log^2 x$ and by Lemma 2.3, $\pi(Q) \leqslant \pi(\sqrt{x}) \leqslant 2.462\sqrt{x}/\log x$. Thus

$$U(x) < \frac{x}{I(x)}\left(\frac{1}{1+1.08/\log x} + 0.156\frac{\log x}{\sqrt{x}}\right) < \frac{x}{I(x)(1+1/\log x)}. \qquad \square$$

Remark. The term $1/\log x$ in Theorem 1 can be increased with more work. Also, a similar result holds if $p-1$ is replaced by $p+a$ for any fixed nonzero a.

3. Small values of $\sigma(\phi(m))$.

In proving Theorem 2, we may restrict our attention to square-free m, since if $p|m$ then

$$\frac{\sigma(\phi(pm))}{pm} = \frac{\sigma(p\phi(m))}{pm} > \frac{\sigma(\phi(m))}{m}.$$

For brevity, write

$$a(m) = \frac{\sigma(\phi(m))}{m}.$$

Throughout, the letter p, with or without subscripts, will always denote a prime. We begin with

$$a(m) = \frac{\sigma(\phi(m))}{\phi(m)} \frac{\phi(m)}{m} = \left(\prod_{p^a \| \phi(m)} \frac{p^{a+1}-1}{p^a(p-1)} \right) \prod_{p|m} \frac{p-1}{p}.$$

Denoting by $e_p(m)$ the exponent of p in the factorization of $\phi(m)$, we have

$$a(m) = \prod_p \frac{p}{p-1}(1 - p^{-1-e_p(m)}) \prod_{p|m} \frac{p-1}{p}. \tag{3.1}$$

We may assume that m has at least 30 distinct prime factors, otherwise if m has k distinct prime factors with $k < 30$ then $2^{k-1}|\phi(m)$. Then (3.1) gives

$$\frac{\sigma(\phi(m))}{m} \geqslant 2(1-2^{-k})\frac{\phi(m)}{m} \geqslant 2(1-2^{-k}) \prod_p \left(1 - \frac{1}{p}\right) \geqslant \frac{1}{10},$$

where the product is over the smallest k primes.

The factors $1 - p^{-1-e_p(m)}$ in (3.1) will likely have product close to 1, so we first make a reduction to this case. Let $S(m)$ be the set of primes dividing $\phi(m)$ (i.e. the set of p with $e_p(m) \geqslant 1$). With a set S of primes fixed, the minimum of $a(m)$ over all m with $S(m) = S$ likely occurs for the "largest" such m.

Lemma 3.1. *If m is squarefree and has at least 30 distinct prime factors, then*

$$a(m) \geqslant \frac{59}{60} \prod_{p \in S} \frac{p}{p-1} \prod_{p \in V(S)} \frac{p-1}{p}, \tag{3.2}$$

where $S = S(m)$ or $S = S(m) \cup \{3\}$, and $V(S)$ is the set of primes p with $p-1$ consisting only of primes is S.

Proof. It suffices to prove (3.2) with $V(S)$ replaced by a subset of $V(S)$, since this only makes the right side larger. Also, since m is divisible by an odd prime, $S(m)$ always contains 2. The basic idea is to multiply m by many primes p which do not divide m, each with $p-1$ having only prime factors in $S(m)$. Then $a(m')$ will not be much larger than $a(m)$, but $e_p(m')$ will be large for most small $p \in S(m)$, making the factors $1 - p^{-1-e_p(m)}$ in (3.1) very close to 1.

We first claim that there is a number m' which has the following properties:

(a) $S(m') = S(m)$,
(b) $a(m) \geqslant 0.98527 a(m')$,
(c) For each prime $3 \leqslant p \leqslant 97$, $p \neq 7$, either $e_p(m') = 0$ or $e_p(m') \geqslant f_p$.

Here f_p is the number of entries corresponding to p in Table 1. For prime p let $q_1(p) < q_2(p) < \cdots$ be the primes with $s(q_i(p)-1) = 2p$. The first f_p of these for $p < 100$ are listed in Table 1. If $e_p(m) > 0$, let Q_p denote the product of the primes $q_i(p)$ listed in the table which do not divide m. If there are none or if $e_p(m) = 0$, let $Q_p = 1$. We take

$$m' = m \prod_{\substack{3\leqslant p\leqslant 97\\ p\neq 7}} Q_p.$$

Clearly (a) and (c) are satisfied with this choice of m'. To show (b), let $m_0 = m$, $m_1 = m_0Q_3$, $m_2 = m_1Q_5$, $\dots$, $m_{23} = m_{22}Q_{97} = m'$. Suppose $1 \leqslant j \leqslant 23$, $p = p_j$, and $e_p(m) \geqslant 1$. Since Q_p is divisible by at least $f_p - e_p(m)$ of the primes $q_i(p)$, (3.1) gives

$$\begin{aligned}\frac{a(m_j)}{a(m_{j-1})} &= \frac{1-2^{-1-e_2(m_j)}}{1-2^{-1-e_2(m_{j-1})}}\frac{1-p^{-1-e_p(m_j)}}{1-p^{-1-e_p(m)}}\prod_{p'|Q_p}(1-1/p') \\ &\leqslant \frac{1-2^{-1-e_2(m_j)}}{1-2^{-1-e_2(m_{j-1})}}\frac{1}{1-p^{-1-e_p(m)}}\prod_{e_p(m)<i\leqslant f_p}(1-1/q_i(p)).\end{aligned} \tag{3.3}$$

If $e_p(m) \geqslant f_p$ the above (empty) product is 1. When $e_p(m) = 0$ then $m_{j-1} = m_j$, so $a(m_{j-1}) = a(m_j)$. Since $e_2(m) \geqslant 29$ by hypothesis, applying (3.3) successively for $j = 1, 2, \dots, 23$ gives

$$\frac{a(m')}{a(m)} \leqslant (1-2^{-30})^{-1}\prod_{\substack{3\leqslant p\leqslant 97\\ p\neq 7}}\max_{2\leqslant h\leqslant f_p+1}(1-p^{-h})^{-1}\prod_{i=h}^{f_p}(1-1/q_i(p)).$$

Here, if $h = f_p + 1$ the (empty) product is 1, which takes care of the case where $e_p(m) = 0$ or $e_p(m) \geqslant f_p$. A short computation now proves (b). In particular, the maximum (over h) occurs at $h = f_p + 1$ for $p = 3, 5, 11, 29, 37, 67, 71, 73, 83$; at $h = 2$ for $p = 13, 17, 19, 23, 31, 41, 43, 47, 59, 61, 79, 89$; at $h = 3$ for $p = 53, 97$.

The next step is to take care of the prime $p = 7$. We show that there is a number m'' satisfying

(d) $S(m'') = S(m')$ or $S(m'') = S(m') \cup \{3\}$,
(e) $a(m') \geqslant (1-2^{-30})a(m'')$,
(f) Either $e_7(m'') = 0$ or $e_7(m'') \geqslant f_7$ with $f_7 = 10$.

If $e_7(m') = 0$ we take $m'' = m'$. If $e_7(m') \geqslant 2$, or $e_7(m') = 1$ and $29 \nmid m$, we take $m'' = Q_7m'$. Then by (3.1), when $e_7(m') \geqslant 2$ we obtain

$$\begin{aligned}\frac{a(m'')}{a(m')} &\leqslant \frac{1}{1-2^{-30}}\max_{3\leqslant h\leqslant f_7+1}(1-7^{-h})^{-1}\prod_{i=h}^{f_7}(1-1/q_i(7)) \\ &\leqslant (1-2^{-30})^{-1},\end{aligned}$$

p	Set of first few $q_i(p)$
3	7, 13, 19, 37, 73, 97, 109, 163, 193, 433, 487, 577, 769, 1153, 1297, 1459, 2593, 2917, 3457, 3889, 10369, 12289, 17497, 18433, 39367, 52489, 139969
5	11, 41, 101, 251, 401, 641, 1601, 4001, 16001, 25601, 40961, 62501, 160001
7	29, 113, 197, 449, 1373, 3137, 50177, 114689, 268913, 470597, 614657
11	23, 89, 353, 1409, 2663, 30977, 170369, 495617, 5767169, 23068673
13	53, 677, 3329, 13313, 35153, 2768897, 13631489
17	137, 157217, 295937, 557057, 1336337
19	1217, 19457, 27437, 7023617, 9904397
23	47, 11777, 33857, 188417, 1557377, 4474457
29	59, 233, 929, 13457, 48779, 59393, 215297
31	7937, 15377, 264017, 458066417
37	149, 593, 5477, 9473, 37889, 151553, 202613, 1401857
41	83, 83969, 6885377, 8821889, 21495809
43	173, 2753, 176129, 1272113, 1893377
47	8837, 2262017,3322337, 36192257
53	107, 1697, 6946817, 46022657
59	1889, 55697, 120833, 410759
61	977, 249857, 56712564737
67	269, 4289, 17957, 287297, 1097729
71	569, 2273, 36353, 80657, 715823
73	293, 4673, 21317, 341057
79	317, 80897, 25563137
83	167, 2657
89	179, 11393, 45569
97	389, 1553, 1589249

TABLE 1. First few primes $q_i(p)$ for $3 \leqslant p \leqslant 97$.

and when $e_7(m') = 1$ and $29 \nmid m$ we have

$$\frac{a(m'')}{a(m')} \leqslant \frac{1 - 7^{-1-f_7}}{1 - 2^{-30}}(49/48)(28/29) \leqslant 1.$$

The last case is when $e_7(m') = 1$ and $29|m$, and we take

$$m'' = Q_3' Q_7 Q_{3,7} m',$$

where $Q_{3,7}$ is the product of the 5 smallest primes q with $s(q-1) = 42$, namely $43, 127, 337, 379, 673$, and Q_3' is the product of the primes $q_i(3)$ which do not divide m'.

If $e_3(m') > 0$, then $e_3(m') \geqslant f_3$ and $Q_3' = 1$ by the construction of m'. Thus

$$\frac{a(m'')}{a(m')} \leqslant \frac{(1 - 3^{-1-f_3})^{-1}}{1 - 2^{-30}} \frac{49}{48} \prod_{i=2}^{f_7}(1 - 1/q_i(7)) \prod_{p|Q_{3,7}} (1 - 1/p) \leqslant 1.$$

Otherwise, $e_3(m')=0$ and

$$\frac{a(m'')}{a(m')} \leqslant \frac{1}{1-2^{-30}}\frac{3}{2}\frac{49}{48}\prod_{i=1}^{f_3}(1-1/q_i(3))\prod_{i=2}^{f_7}(1-1/q_i(7))\prod_{p|Q_{3,7}}(1-1/p) \leqslant 1.$$

Putting together (a)-(f), we have

$$\begin{aligned} a(m) &\geqslant 0.98527a(m') \geqslant (1-2^{-30})0.98527a(m'') \\ &\geqslant 0.98526 \prod_{p\in S(m'')}\frac{p}{p-1}\prod_{3\leqslant p\leqslant 97}(1-p^{-1-f(p)})\prod_{p\geqslant 101}(1-p^{-2})\prod_{p|m''}\frac{p-1}{p} \\ &\geqslant 0.98348 \prod_{p\in S(m'')}\frac{p}{p-1}\prod_{p\in V(S(m''))}\frac{p-1}{p}. \qquad \square \end{aligned}$$

With Lemma 3.1 we have essentially reduced the problem to finding a lower bound for $\inf_S E(S)$, where

$$E(S) := \sum_{p\in S}\log(p/(p-1)) - \sum_{p\in V(S)}\log(p/(p-1)), \tag{3.4}$$

under the assumption that $2\in S$.

To bound $E(S)$, first define

$$\alpha(u) = -\log(H(e^{e^u})) = \sum_{\substack{p\leqslant e^{e^u}\\ p\in S}}\log(p/(p-1)).$$

Let W be the least real number $\geqslant 10$ for which

$$\int_1^{\sqrt{W}}\frac{H(t)\log t}{t}\,dt = \log W. \tag{3.5}$$

Such W exists because the left side is continuous in W and $H(t)$ is constant for large t. By (2.2),

$$W \geqslant 7.872\cdot 10^6, \qquad \omega := \log\log W \geqslant 2.7649. \tag{3.6}$$

For $x<W$, $I(x)<\log x$ and so the right side in Theorem 1 is $>(1+o(1))x/\log x > (1+o(1))\pi(x)$, which is the trivial bound. Let

$$B = \alpha(\omega-\log 2) = -\log H(\sqrt{W}).$$

We next show that

$$\frac{e^\omega}{8} < e^B < \frac{e^\omega - 3.37}{4}. \tag{3.7}$$

By (3.5),

$$e^\omega > H(\sqrt{W}) \int_1^{\sqrt{W}} \frac{\log t}{t}\, dt = \frac{e^{2\omega - B}}{8},$$

which gives the lower bound in (3.7). Next, applying (2.8) together with (2.2) and (3.6) gives

$$\begin{aligned} e^\omega &\leqslant H(\sqrt{W}) \int_{66}^{\sqrt{W}} \frac{\log\sqrt{W}}{t} \left(1 + \frac{1}{\log^2\sqrt{W}}\right)\left(1 + \frac{1}{\log^2 t}\right) dt + \int_1^{66} \frac{H(t)\log t}{t}\, dt \\ &\leqslant e^{-B}(L + 1/L)(L - 1/L - \log 66 + 1/\log 66) + \frac{4\log^2 66 + 1.922}{16}, \quad (L = \log\sqrt{W}) \\ &\leqslant e^{\omega - B}\left(\frac{e^\omega}{4} - 1.97\right) + 4.51. \end{aligned}$$

Therefore,

$$e^B \leqslant \frac{e^\omega}{4} \frac{1 - 7.88e^{-\omega}}{1 - 4.51e^{-\omega}} = \frac{e^\omega}{4}\left(1 - \frac{3.37}{e^\omega - 4.51}\right),$$

and the upper bound in (3.7) follows. Denote by P the largest prime in S, let $K = \log\log P$, and let

$$C = \sum_{p \in S} \log(p/(p-1)) = \lim_{u\to\infty} \alpha(u).$$

Set $\delta = 8e^{-2\omega}$. By (2.8),

$$\alpha(u) \leqslant \begin{cases} B + u - (\omega - \log 2) + \delta & (\omega - \log 2 \leqslant u \leqslant C - B + \omega - \log 2) \\ C & (u > C - B + \omega - \log 2). \end{cases} \tag{3.8}$$

By partial summation, Theorem 1, (3.5) and the bound $(t-1)(1 + 1/\log t) \geqslant t$, we have

$$\begin{aligned} T := -\sum_{p \in V(S), p > W} \log(1 - 1/p) &= U(W)\log(1 - 1/W) + \int_W^\infty \frac{U(t)}{t^2 - t}\, dt \\ &\leqslant -\frac{U(W)}{W} + \int_W^\infty \frac{dt}{tI(t)} \\ &= -\frac{U(W)}{W} + \int_\omega^\infty \frac{e^v}{I\left(e^{e^v}\right)}\, dv \\ &= -\frac{U(W)}{W} + \int_\omega^\infty e^v \left(\int_{-\infty}^{v - \log 2} e^{2z - \alpha(z)}\, dz\right)^{-1} dv \\ &= -\frac{U(W)}{W} + \int_\omega^\infty e^v \left(e^\omega + \int_{\omega - \log 2}^{v - \log 2} e^{2z - \alpha(z)}\, dz\right)^{-1} dv. \end{aligned}$$

First, consider the case where $K \leqslant \omega - \log 2$, i.e. $B = C$. Then $\alpha(u) = B$ for $u \geqslant \omega - \log 2$, so by (3.7)

$$\begin{aligned} T &\leqslant -\frac{U(W)}{W} + \int_\omega^\infty e^v \left(e^\omega + \tfrac{1}{8}e^{-B}(e^{2v} - e^{2\omega})\right)^{-1} dv \\ &= -\frac{U(W)}{W} + 8e^{B-\omega}\frac{\arctan \xi}{\xi}, \quad \xi = \sqrt{8e^{B-\omega} - 1} \in (0,1). \end{aligned}$$

Therefore,

$$E(S) \geqslant B + \frac{U(W)}{W} + \sum_{p \in V(S), p \leqslant W} \log(1 - 1/p) - 8e^{B-\omega}\frac{\arctan \xi}{\xi}.$$

By Lemmas 2.2 and 2.3,

$$\begin{aligned} \frac{U(W)}{W} + \sum_{p \in V(S), p \leqslant W} \log(1 - 1/p) &= \sum_{\substack{p \in V(S) \\ p \leqslant W}} (\log(1 - 1/p) + 1/W) \\ &\geqslant \sum_{p \leqslant W} (\log(1 - 1/p) + 1/W) \\ &\geqslant \frac{1}{\log W} + \log\left(\frac{e^{-\gamma}}{\log W}\left(1 + \frac{1}{\log^2 W}\right)^{-1}\right) \qquad (3.9) \\ &\geqslant -\gamma - \omega + \frac{1}{\log W} - \frac{1}{\log^2 W} \\ &\geqslant -\gamma - \omega. \end{aligned}$$

Write $e^B = \lambda e^\omega$, so that $\lambda \in (1/8, 1/4)$ by (3.7). Then

$$E(S) \geqslant \log \lambda - \gamma - 8\lambda \frac{\arctan \sqrt{8\lambda - 1}}{\sqrt{8\lambda - 1}}.$$

Now $\frac{\arctan(x)}{x} \leqslant 1 - \frac{1}{5}x^2$ for $0 \leqslant x \leqslant 1$, so that $E(S) \geqslant \log \lambda - \gamma + 12.8\lambda^2 - 9.6\lambda$. The right side is increasing in λ, so that

$$E(S) \geqslant -\log 8 - \gamma - 1. \qquad (3.10)$$

In the case where $K > \omega - \log 2$, let $\eta = C - B + \omega$. By (3.9),

$$\begin{aligned} T &\leqslant -\frac{U(W)}{W} + I_1 + I_2, \\ I_1 &= \int_\omega^\eta e^v \left(e^\omega + \int_{\omega - \log 2}^{v - \log 2} e^{z - B + \omega - \log 2 - \delta}\, dz\right)^{-1} dv \\ &= \int_\omega^\eta e^v \left(e^\omega + \tfrac{1}{4}e^{\omega - B - \delta}(e^v - e^\omega)\right)^{-1} dv, \\ I_2 &= \int_\eta^\infty e^v \left(e^\omega + \tfrac{1}{4}e^{\omega - B - \delta}(e^\eta - e^\omega) + \int_{\eta - \log 2}^{v - \log 2} e^{2z - C}\, dz\right)^{-1} dv. \end{aligned}$$

Then

$$
\begin{aligned}
I_1 &= 4e^{B-\omega+\delta} \int_{e^\omega}^{e^\eta} \frac{dz}{z - e^\omega + 4e^{B+\delta}} \\
&= 4e^{B-\omega+\delta} \log\left(\frac{e^\eta - e^\omega}{4e^{B+\delta}} + 1 \right)
\end{aligned}
$$

and

$$
I_2 = 8e^C \int_{e^\eta}^{\infty} \frac{dz}{z^2 + 8e^{C+\omega} + e^{2\eta}(2e^{-\delta} - 1) - 2e^{\omega+\eta-\delta}}.
$$

Again set $\lambda = e^{B-\omega} \in (1/8, 1/4)$. Also define $\beta = e^{C-B} > 1$ and $\xi = (8\lambda - 2e^{-\delta})/\beta + 2e^{-\delta} - 1 > 0$. The last inequality follows from $2e^{-\delta} - 1 > 0$. Therefore, by (3.9),

$$
\begin{aligned}
E(S) &\geqslant C - \gamma - \omega - 4e^\delta \lambda \log\left(1 + \frac{\beta - 1}{4\lambda e^\delta}\right) + 8\lambda \int_1^\infty \frac{dy}{y^2 + \xi} \\
&= \log\lambda + \log\beta - \gamma - 4e^\delta\lambda \log\left(1 + \frac{\beta-1}{4\lambda e^\delta}\right) + 8\lambda \frac{\arctan\sqrt{\xi}}{\sqrt{\xi}}.
\end{aligned}
\tag{3.11}
$$

By (3.7), $\lambda \leqslant \frac{1}{4}(1 - 3.37e^{-\omega}) \leqslant \frac{1}{4}(1-\delta) \leqslant e^{-\delta}/4$, so $4\lambda e^\delta < 1$. In the region $1 \leqslant \beta$, $\frac{1}{8} \leqslant \lambda \leqslant \frac{e^{-\delta}}{4}$, the right side of (3.11) is minimum at $\beta = 1$, $\lambda = \frac{1}{8}$. Therefore (3.10) follows in this case as well. Theorem 2 now follows from Lemma 3.1 and (3.10).

4. A heuristic

From the preceding argument, we obtain the worst bound for $E(S)$ when the bound for $\alpha(u)$ given by (3.8) is sharp, i.e. when S consists of 2 plus all the primes in an interval $(y, z]$. Thus, it is reasonable that taking S to be the set of primes less than y, say, will produce small values of $E(S)$. Then $V = V(S)$ consists of all primes p such that all prime factors of $p-1$ are $\leqslant y$. Let $\Psi(x,y)$ denote the number of integers $n \leqslant x$, all of whose prime factors are $\leqslant y$. Then we expect that $U(x) \approx \Psi(x,y)/\log x$. It is known that in a wide range of x, y (see [HT]) that $\Psi(x,y) \sim x\rho(u)$, where $u = \frac{\log x}{\log y}$ and ρ is the Dickman-de Bruijn function defined by

$$
\rho(u) = 1 \quad (0 \leqslant u \leqslant 1), \qquad \rho(u) = 1 - \int_1^u \frac{\rho(v-1)}{v}\, dv \quad (u > 1).
$$

Then, arguing heuristically,

$$\begin{aligned} E(S) &= \sum_{p>y, p\in V(S)} \log\left(\frac{p-1}{p}\right) \\ &= -U(y)\log(1-1/y) - \int_y^\infty \frac{U(t)}{t^2-t}\,dt \\ &\approx \frac{\pi(y)}{y} - \int_y^\infty \frac{\rho(\log t/\log y)}{t\log t}\,dt \\ &\approx \frac{1}{\log y} - \int_1^\infty \frac{\rho(u)}{u}\,du \\ &\approx \frac{1}{\log y} - 0.5219. \end{aligned}$$

Thus, we expect $\exp E(S) \geqslant e^{-0.5219} \geqslant 0.593$ for such S when y is large.

Below we provide a table of rigorous lower bounds for $e^{E(S)}$ for various sets S. The values of $e^{E(S)}$ listed are truncated in the third decimal place.

S	$e^{E(S)}$	S	$e^{E(S)}$	S	$e^{E(S)}$
2	0.500	2, 3, 5	0.518	2, 3, 5, 7	0.529
2, 3	0.517	2, 3, 7	0.547	2, 3, 5, 11	0.517
2, 5	0.543	2, 3, 11	0.520	2, 3, 7, 11	0.548
2, 7	0.553	2, 3, 13	0.534	2, 3, 11, 29	0.523
2, 11	0.518	2, 3, 17	0.534	2, 3, 13, 29	0.537
2, 13	0.530	2, 5, 7	0.588	2, 3, 5, 11, 29	0.518
2, 17	0.527	2, 5, 11	0.562	2, 3, 5, 7, 11	0.524
2, 19	0.527	2, 7, 11	0.572	2, 3, 11, 23, 29	0.526
2, 23	0.511	2, 11, 29	0.524	2, 3, 5, 11, 23	0.517
2, 29	0.506	2, 13, 29	0.537		

TABLE 2. Values of $e^{E(S)}$.

For certain classes of small S, it is easy to prove that $e^{E(S)} > \frac{1}{2}$.

Lemma 4.1. *For all primes $p \geqslant 5$,*

$$e^{E(\{2,p\})} \geqslant \frac{1}{2}\left(1 + \frac{1}{3p+1}\right).$$

Proof. For $p = 5, 7$ this follows from Table 1. Suppose $p \geqslant 11$. If $2^h + 1$ is prime, then h is a power of 2. If $p \equiv 2 \pmod 3$, then $m = 2^k p^l + 1$ is divisible by 3 whenever $k+l$ is odd, hence m is composite. Therefore

$$\begin{aligned} e^{E(S)} &\geqslant \frac{p}{p-1} \prod_{h=0}^{\infty} \frac{2^{2^h}}{2^{2^h}+1} \prod_{\substack{l+k \text{ odd} \\ l\geqslant 1, k\geqslant 1}} \left(1 - \frac{1}{2^k p^l + 1}\right) \\ &\geqslant \frac{p}{2p-2} \exp\left\{ - \sum_{\substack{l+k \text{ odd} \\ l\geqslant 1, k\geqslant 1}} \frac{1}{2^k p^l} \right\} \\ &= \frac{p}{2p-2} \exp\left(- \frac{2p+1}{3(p^2-1)} \right) \\ &\geqslant \frac{p}{2p-2} \left(1 - \frac{2p+1}{3(p^2-1)} \right) \\ &\geqslant \frac{1}{2} \left(1 + \frac{1}{3p+1} \right). \end{aligned}$$

When $p \equiv 1 \pmod 3$, then $2^k p^l + 1$ is divisible by 3 whenever $k+l$ is even, and a similar argument gives a stronger bound. □

Similarly, it can be shown that $e^{E(\{2,3,p\})} \geqslant 0.51$ for all prime $p \geqslant 5$.

One may attempt to **disprove** (1.1) by a certain explicit construction, adding primes successively to S, where at each stage $E(S)$ decreases. For example, $E(\{2,3,5,11\} < E(\{2,3,5\})$. Also, if p is prime and the numbers $kp+1$ ($k=2$, 6, 8, 12, 18, 32, 36, 48, 72, 96) are all prime (in particular $p \equiv 86 \pmod{105}$) then

$$e^{E(\{2,3,p\})} \leqslant e^{E(\{2,3\})} \left(1 - \frac{0.034}{p}\right).$$

However, it looks hopeless to disprove (1.1) in this way.

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