

DIFFERENCES BETWEEN TOTIENTS

KEVIN FORD AND SERGEI KONYAGIN

ABSTRACT. We study the set \mathcal{D} of positive integers d for which the equation $\phi(a) - \phi(b) = d$ has infinitely many solution pairs (a, b) . We show that $\min \mathcal{D} \leq 154$, exhibit a specific A so that every multiple of A is in \mathcal{D} , and show that any progression $a \pmod d$ with $4|a$ and $4|d$, contains infinitely many elements of \mathcal{D} . We also show that the Generalized Elliott-Halberstam Conjecture, as defined in [6], implies that \mathcal{D} contains all positive, even integers.

1. INTRODUCTION

Let $\mathcal{V} = \{v_1, v_2, \dots\}$ be the set of totients, that is, \mathcal{V} is the image of Euler's totient function $\phi(n)$. In this paper we study the set \mathcal{D} of positive integers which are infinitely often a difference of two elements of \mathcal{V} . A classical conjecture asserts that every even positive integer is infinitely often the difference of two primes, and this implies immediately that \mathcal{D} is the set of all positive, even integers. We are interested in what can be accomplished unconditionally, by leveraging the recent breakthroughs on gaps between consecutive primes by Zhang [8], Maynard [4], Tao (unpublished) and the PolyMath8b project [6]. We let \mathcal{E} be the set of positive even numbers that are infinitely often the difference of two primes. Clearly $\mathcal{E} \subseteq \mathcal{D}$. In this note we prove some results about \mathcal{D} which are not known for \mathcal{E} .

The behavior of the smallest elements of \mathcal{D} arose in recent work of Fouvry and Waldschmidt [2] concerning representation of integers by cyclotomic forms, and the problem of studying the differences of totients was also posted in a list of open problems by Shparlinski [7, Problem 56]. Our paper is a companion of the recent work of the first author [1] concerning the equation $\phi(n+k) = \phi(n)$ for fixed k .

It is known [6] that $\min \mathcal{E} \leq 246$ and thus $\min \mathcal{D} \leq 246$. We can do somewhat better.

Theorem 1. *We have $\min \mathcal{D} \leq 154$.*

Although there is no specific even integer which is known to be infinitely often the difference of two primes, we give an infinite family of specific numbers that are in \mathcal{D} .

Theorem 2. *Let $a_0 = \prod_{p \leq 47} p$ and $b_0 = \text{lcm}[1, 2, \dots, 49]$. Then every multiple of $\phi(a_0 b_0) a_0$ lies in \mathcal{D} .*

Granville, Kane, Koukoulopoulos and Lemke-Oliver [3] showed that \mathcal{E} has lower asymptotic density at least $\frac{1}{354}$ and thus so does \mathcal{D} . We do not know how to prove a better lower bound for the density and leave this as an open problem.

Date: September 16, 2021.

2010 MSC 11A25, 11N64 (primary), 11D85 (secondary).

First author supported by National Science Foundation Grant DMS-1802139.

Keywords and phrases: totients, prime gaps, admissible sets.

Central to the works [4, 5, 6, 8] is the concept of an admissible set of linear forms. For positive integers a_i and integers b_i , the set of affine-linear forms $a_1x + b_1, \dots, a_kx + b_k$ is admissible if, for every prime p , there is an $x \in \mathbb{Z}$ such that $p \nmid (a_1x + b_1) \cdots (a_kx + b_k)$.

Definition. Hypothesis DHL[k, m] is the statement that for any admissible k -tuple of linear forms $a_i n + b_i$, $1 \leq i \leq k$, for infinitely many n , at least m of them are simultaneously prime.

In this paper we are concerned with the statements DHL[$k, 2$]. The Polymath8b project [6], plus subsequent work of Maynard [5], established DHL[50, 2] unconditionally.

The Elliott-Halberstam Conjecture implies DHL[5, 2], see [4]. The Generalized Elliott-Halberstam Conjecture implies DHL[3, 2] (see [6] for details).

Theorem 3. *We have*

- (i) DHL[3, 2] implies that $\mathcal{D} = \{2, 4, 6, 8, 10, \dots\}$, the set of all positive even integers.
- (ii) DHL[4, 2] implies that \mathcal{D} contains every positive multiple of 4.
- (iii) DHL[5, 2] implies that $\min \mathcal{D} \leq 6$.
- (iv) DHL[6, 2] implies $8 \in \mathcal{D}$.

By contrast, for any $k \geq 2$, DHL[$k, 2$] implies that $\liminf p_{n+1} - p_n \leq a_k$, where a_k is the minimum of $h_k - h_1$ over all admissible k -tuples $n + h_1, \dots, n + h_k$. We have $a_3 = 6$, $a_4 = 8$, $a_5 = 12$ and $a_6 = 16$.

We show parts of Theorem 3 (ii) and (iii) using a more general result.

Theorem 4. *Assume DHL[$k, 2$], with $k \geq 3$. Also assume that there are integers $1 < m_1 < \dots < m_k$ and $\ell_{i,j}$ for $1 \leq i < j \leq k$ and such that*

$$\frac{\ell_{i,j} m_i}{m_j - m_i} \in \mathcal{V}, \quad \frac{\ell_{i,j} m_j}{m_j - m_i} \in \mathcal{V} \quad (1 \leq i < j \leq k).$$

Then $L \leq 2 \max \ell_{i,j}$. Moreover, if $\ell_{i,j} = \ell$ for all i, j then $2\ell \in \mathcal{D}$.

In Section 3, we give a heuristic argument that there exist numbers m_1, \dots, m_{50} satisfying the hypothesis of Theorem 4 with $\ell_{i,j} = 2$ for all $i < j$. In this case we achieve an unconditional proof that $4 \in \mathcal{D}$. Actually finding such m_i seems computationally difficult, however.

Does every arithmetic progression $a \pmod d$ containing even numbers have infinitely many elements of \mathcal{D} ? We answer in the affirmative if $4|d$ and $4|a$. The case $a \equiv 2 \pmod 4$ is more difficult; see our Remarks following the proof in Section 4.

Theorem 5. *Let a, d be positive integers with $4|a$, $4|d$. Then the progressions $a \pmod d$ contains infinitely many elements of \mathcal{D} .*

Observe that, even assuming DHL[3, 2], there is no specific progression $a \pmod d$, not containing 0, which is known to contain a number that is infinitely often the difference of two primes.

When $4|d$ and $a \equiv 2 \pmod 4$ we can sometimes show that \mathcal{D} contains infinitely many elements that are $\equiv a \pmod d$; see the Remarks following the proof of Theorem 5 in Section 4.

2. PROOF OF THEOREMS 1–4

Proof of Theorem 1. Define

$$\begin{aligned} S_1 &= \{41, 43, 47, 53, 67, 71\}, \\ S_2 &= \{59, 61, 67, 71, 73, 83, 89, 101, 103, 107, 109, 113, 127, 131, 137, 139\}, \\ S_4 &= \{p \text{ prime} : 127 \leq p \leq 271\}. \end{aligned}$$

and consider the collection of 50 linear forms

$$n + a \quad (a \in S_1), \quad 2n + a \quad (a \in S_2), \quad 4n + a \quad (n \in S_4).$$

This collection is admissible; indeed if $p < 41$ and $n = 0$ then all of them are coprime to p . For $p > 50$ it is clear that there is an n for which all of them are coprime to p . When $p \in \{41, 43, 47\}$ we take $n = 1, 3, 8$, respectively, and then all of the forms are coprime to p . By DHL[50, 2], there are two of these forms that are simultaneously prime for infinitely many n . If both forms are of the type $n + a$ for $a \in S_1$, then this shows that $\min \mathcal{D} \leq 71 - 41 = 30$. Likewise, if both forms are of the type $2n + a$ for $a \in S_2$ then $\min \mathcal{D} \leq 139 - 59 = 80$ and if both forms are of the type $4n + a$ where $a \in S_4$, then $\min \mathcal{D} \leq 271 - 127 = 144$. Now suppose for infinitely many n , $n + a$ and $2n + b$ are both prime, where $a \in S_1$, $b \in S_2$. Then

$$\phi(4(n + a)) = 2n + 2a - 2, \quad \phi(2n + b) = 2n + b - 1,$$

which shows that $|b - 2a + 1| \in \mathcal{D}$. We have $b - 2a + 1 \neq 0$ for all choices, and the maximum of $|b - 2a + 1|$ is 82, and hence $\min \mathcal{D} \leq 82$. Similarly, if for infinitely many n , $2n + a$ and $4n + b$ are both prime, where $a \in S_2$, $b \in S_4$, then $|b - 2a + 1| \in \mathcal{D}$. Hence $\min \mathcal{D} \leq 154$. Finally, if for infinitely many n , $n + a$ and $4n + b$ are both prime, where $a \in S_1$, $b \in S_4$, then $|b - 4a + 3| \in \mathcal{D}$ since

$$\phi(8(n + a)) = 4(n + a - 1), \quad \phi(4n + b) = 4n + b - 1.$$

In all cases $0 < |b - 4a + 3| \leq 154$. □

Proof of Theorem 2. Let

$$a_0 = \prod_{p \leq 47} p, \quad b_0 = \text{lcm}[1, 2, \dots, 49].$$

Let $k \in \mathbb{N}$ and consider the admissible set of linear forms $n + ka_0, n + 2ka_0, \dots, n + 50ka_0$. Since DHL[50, 2] holds, for any $k \in \mathbb{N}$ there exists $j_k \in \{1, \dots, 49\}$ such that the equation

$$\phi(u) - \phi(v) = u - v = kj_k a_0$$

has infinitely many solutions in primes u, v . Since every prime factor of j_k is a divisor of $a_0 b_0 / j_k$, we have $\phi(a_0 b_0 l / j_k) = \phi(a_0 b_0 l) / j_k$ for any $l \in \mathbb{N}$. Therefore,

$$\phi(a_0 b_0 u / j_k) - \phi(a_0 b_0 v / j_k) = \phi(a_0 b_0) (\phi(u) - \phi(v)) / j_k = \phi(a_0 b_0) a_0 k,$$

as required. □

Proof of Theorem 4. The set of forms m_1n-1, \dots, m_kn-1 is clearly admissible. By DHL[$k, 2$], for some pair $i < j$ and for infinitely many n , $m_in - 1$ and $m_jn - 1$ are prime. Let $\ell = \ell_{i,j}$ and suppose that x, y satisfy

$$\phi(x) = \frac{\ell m_i}{m_j - m_i}, \quad \phi(y) = \frac{\ell m_j}{m_j - m_i}.$$

Then for sufficiently large n

$$\phi(x(m_jn - 1)) - \phi(y(m_in - 1)) = \frac{-2\ell m_i + 2\ell m_j}{m_j - m_i} = 2\ell. \quad \square$$

Proof of Theorem 3. (i) Let $h \in \mathbb{N}$ and consider the triple $\{n+1, n+2h+1, 2n+2h+1\}$. This is admissible, since when $n = 0$, all of the forms are odd, and similarly none are divisible by 3 for some $n \in \{0, 1\}$. By DHL[3, 2], either (i) $n+1$ and $n+2h+1$ are infinitely often both prime, (ii) $n+1$ and $2n+2h+1$ are infinitely often both prime or (iii) $n+2h+1$ and $2n+2h+1$ are infinitely often both prime. In case (i) we have $\phi(n+2h+1) - \phi(n+1) = 2h$, in case (ii) we have $\phi(2n+2h+1) - \phi(4(n+1)) = 2h$, and in case (iii) we have $\phi(4(n+2h+1)) - \phi(2n+2h+1) = 2h$.

(ii) The deduction $4 \in \mathcal{D}$ follows from Theorem 4 using $\ell_{i,j} = 2$ for all i, j and

$$\{m_1, \dots, m_4\} = \{6, 8, 9, 12\}.$$

Now suppose that d is congruent to 0 or 4 modulo 12, and define a by $d = 2(a+1)$. In particular, $(a, 6) = 1$. Thus, the set of forms $m_in - a$, $1 \leq i \leq 4$, are admissible. By DHL[4, 2], for some pair $i < j$ and for infinitely many n , $m_in - a$ and $m_jn - a$ are prime. Suppose that x, y satisfy

$$\phi(x) = \frac{2m_i}{m_j - m_i}, \quad \phi(y) = \frac{2m_j}{m_j - m_i}.$$

Then for sufficiently large n

$$\phi(x(m_jn - a)) - \phi(y(m_in - a)) = 2(a+1) = d. \quad \square$$

Hence, $d \in \mathcal{D}$.

Finally, if $d \equiv 8 \pmod{12}$, write $d = 2(b-1)$, so that $(b, 6) = 1$. Similarly, the set of forms $m_in + b$, $1 \leq i \leq 4$, are admissible and we conclude that $d \in \mathcal{D}$.

(iii) Consider the admissible set of forms $\{f_1(n), \dots, f_5(n)\} = \{n, n+2, 2n+1, 4n-1, 4n+3\}$. Indeed, if $n \equiv 11 \pmod{30}$ then all of the forms are coprime to 30. By DHL[5, 2], for some $i < j$ and infinitely many n , $f_i(n)$ and $f_j(n)$ are both prime. Say $f_i(n) = an + b$ and $f_j(n) = cn + d$ with $c/a \in \{1, 2, 4\}$. Then

$$\phi((2c/a)(an + b)) = (c/a)(an + b - 1) = cn + (c/a)(b - 1)$$

and

$$\phi(cn + d) = cn + d - 1.$$

Thus, $|(c/a)(b-1) - (d-1)| \in \mathcal{D}$. In all cases, $|(c/a)(b-1) - (d-1)| \leq 6$.

(iv) Use Theorem 4 with the set

$$\{h - 72, h - 66, h - 64, h - 63, h - 60, h\}, \quad h = 120193920,$$

and $\ell_{i,j} = 4$ for all i, j . We used PARI/GP to verify that $4m_i/(m_j - m_i) \in \mathcal{V}$ and $4m_j/(m_j - m_i) \in \mathcal{V}$ for all $i < j$.

3. A HEURISTIC ARGUMENT

In this section, we give an argument that there should exist m_1, \dots, m_{50} satisfying the hypothesis of Theorem 4 with $\ell_{i,j} = 2$ for all i, j . We first give a general construction of numbers with $\frac{m_i}{m_j - m_i}$ all integers.

Lemma 3.1. *For any positive integer b and $k \geq 2$ there is a set $\{m_1, \dots, m_k\}$ of positive integers with $m_1 < m_2 < \dots < m_k$ and with*

$$(3.1) \quad b \mid \frac{m_j}{m_j - m_i} \quad (1 \leq i < j \leq k).$$

Proof. Induction on k . When $k = 2$ take $\{m_1, m_2\} = \{2b - 1, 2b\}$. Now assume (3.1) holds for some $k \geq 2$. Let M be the least common multiple of the $\binom{k}{2}$ numbers

$$m_j - m_i \quad (1 \leq i < j \leq k),$$

and let K be the least common multiple of the numbers

$$M, bM - m_1, \dots, bM - m_k.$$

We claim that the set

$$\{m'_1, \dots, m'_{k+1}\} = \{Kb - bM + m_1, \dots, Kb - bM + m_k, Kb\}$$

satisfies (3.1). Indeed, when $1 \leq i < j \leq k$ we have

$$\frac{m'_j}{m'_j - m'_i} = \frac{Kb - bM + m_j}{m_j - m_i},$$

which by hypothesis is divisible by b . Finally, for any $i \leq k$,

$$\frac{m'_{k+1}}{m'_{k+1} - m'_i} = \frac{Kb}{bM - m_i},$$

which is also divisible by b . □

Now let $b = \prod_{p \leq 2450} p$, and apply Lemma 3.1 with $k = 50$. There is a set $\{m_1, \dots, m_{50}\}$ such that for all $i < j$,

$$(3.2) \quad b \mid \frac{m_j}{m_j - m_i}.$$

Let M be the least common multiple of the $\binom{50}{2}$ numbers

$$m_j - m_i \quad (1 \leq i < j \leq 50).$$

Then for any $h \in \mathbb{N}$, the set $\{m_1 + hbM, \dots, m_{50} + hbM\}$ has the same property (3.2). The collection of 2450 linear forms (in h)

$$\frac{2(m_i + hbM)}{m_j - m_i} + 1 = \frac{2(m_j + hbM)}{m_j - m_i} - 1 \quad (1 \leq i < j \leq 50)$$

and

$$\frac{2(m_j + hbM)}{m_j - m_i} + 1 \quad (1 \leq i < j \leq 50)$$

is admissible by (3.2), and the Prime k -tuples conjecture implies that all of these are prime for some h . We need only the existence of one h , and then the hypotheses of Theorem 4 hold with $\ell_{i,j} = 2$ for all i, j , and consequently $4 \in \mathcal{D}$. Discovering such an h , however, appears to be computationally infeasible.

4. TOTIENT GAPS IN PROGRESSIONS: PROOF OF THEOREM 5

Lemma 4.1. *Suppose that $D \in \mathbb{N}$ and $4|a$. Then there exist v_1 and v_2 such that $(D, v_1) = (D, v_2) = 1$ and $(v_1 - 1)(v_2 - 1) \equiv a \pmod{D}$ or $(v_1 + 1)(v_2 - 1) \equiv a \pmod{D}$.*

Proof. We use the Chinese Remainder Theorem. We will prove that if $D = p^\alpha$ is a prime power, then for $p \neq 3$ we can find a pair v_1 and v_2 such that $(D, v_1) = (D, v_2) = 1$ and $(v_1 - 1)(v_2 - 1) \equiv a \pmod{D}$ and another pair v'_1 and v'_2 such that $(D, v'_1) = (D, v'_2) = 1$ and $(v'_1 + 1)(v'_2 - 1) \equiv a \pmod{D}$. If $p = 3$ then we will find appropriate v_1 and v_2 such that one of desired congruences hold. This will suffice for the proof of the lemma.

If $p \neq 3$ and for any a , $4|a$, a pair v_1 and v_2 exists such that $(D, v_1) = (D, v_2) = 1$ and $(v_1 - 1)(v_2 - 1) \equiv a \pmod{D}$, then there also exists a pair v'_1 and v'_2 as well. Indeed, take any possible D and a . Then, by our supposition, there are v_1 and v_2 such that $(D, v_1) = (D, v_2) = 1$ and $(v_1 - 1)(v_2 - 1) \equiv -a \pmod{D}$. Then for $v'_1 = -v_1$ and $v'_2 = v_2$ the desired congruence $(v'_1 + 1)(v'_2 - 1) \equiv a \pmod{D}$ holds.

Consider $p = 2$. Take $v_1 \equiv 3 \pmod{p^\alpha}$ and $v_2 \equiv \frac{a}{2} + 1 \pmod{p^\alpha}$. Then $(v_1 - 1)(v_2 - 1) \equiv a \pmod{p^\alpha}$ and both v_1, v_2 are odd.

Consider $p = 3$. If $a \equiv 0 \pmod{3}$ or $a \equiv 1 \pmod{3}$, let $v_1 \equiv 2 \pmod{3^\alpha}$ and $v_2 \equiv a + 1 \pmod{3^\alpha}$. Then we have $(v_1 - 1)(v_2 - 1) \equiv a \pmod{3^\alpha}$. If $a \equiv -1 \pmod{3}$ let $v_1 \equiv -2 \pmod{3^\alpha}$ and $v_2 \equiv -a + 1 \pmod{3^\alpha}$. Then we have $(v_1 + 1)(v_2 - 1) \equiv a \pmod{3^\alpha}$.

Consider $p > 3$. Take v_2 so that $v_2 \notin \{0, 1, 1 - a\} \pmod{p}$. Then there is some $v_1 \not\equiv 0 \pmod{p}$ such that $(v_2 - 1)(v_1 - 1) \equiv a \pmod{p^\alpha}$. \square

Proof of Theorem 5. Let D be any positive integer satisfying

- (a) $d|D$;
- (b) D, D^2, \dots, D^{49} are all in \mathcal{V} .

For example, let P be the largest prime factor of d , γ sufficiently large and

$$D = d \prod_{p \leq P} p^\gamma.$$

Indeed, if

$$D = \prod_{p \leq P} p^{\alpha(p)}, \quad \prod_{p \leq P} (p - 1) = \prod_{p \leq P} p^{\beta(p)},$$

then, assuming $\gamma \geq \max \beta(p)$, for all $j \geq 1$ we have

$$\phi \left(\prod_{p \leq P} p^{j\alpha(p) - \beta(p) + 1} \right) = D^j.$$

Now take any D satisfying (a) and (b) above, and let v be coprime to D . Then the set

$$f_j(x) = D^j x - v, \quad j = 1, \dots, 50,$$

of linear forms is admissible. Indeed, if $p \nmid v$, then $f_1(0) \cdots f_{50}(0) = v^{50} \not\equiv 0 \pmod{p}$, and if $p|v$ then $p \nmid D$ and $f_1(1) \cdots f_k(1) \equiv D^{1225} \not\equiv 0 \pmod{p}$. Since DHL[50, 2] holds, there are $j_1 < j_2$ such that for infinitely many positive integers x both numbers $p_1 = D^{j_1}x - v$, $p_2 = D^{j_2}x - v$ are primes. Denote $j = j_2 - j_1$. There exists l such that $\phi(l) = D^j$. If x is large enough, then $(p_1, l) = (p_2, l) = 1$. We have

$$(4.1) \quad \phi(p_2) - \phi(p_1l) = (p_2 - 1) - (p_1 - 1)D^j = (v + 1)(D^j - 1).$$

Let v_1, v_2 be as in Lemma 4.1, and let v satisfy

$$\begin{cases} v \equiv -v_1 \pmod{D}, v > 0 & \text{if } (v_1 - 1)(v_2 - 1) \equiv a \pmod{D}, \\ v \equiv v_1 \pmod{D}, v < -1 & \text{otherwise.} \end{cases}$$

Fix a prime $q \equiv v_2 \pmod{D}$ with $(q, l) = 1$, and assume that $p_1, p_2 > q$. Then

$$(4.2) \quad \phi(p_2q) - \phi(p_1lq) = (q - 1)(v + 1)(D^j - 1).$$

Thus, $|(q-1)(v+1)(D^j-1)| \in \mathcal{D}$. The right side of (4.2) is $\equiv (q-1)(-v-1) \equiv (v_2-1)(-v-1) \pmod{D}$ and has sign equal to the sign of v . If $(v_1 - 1)(v_2 - 1) \equiv a \pmod{D}$, then $v > 0$ and thus the right side of (4.2) is positive and congruent to $(v_2 - 1)(v_1 - 1) \equiv a \pmod{D}$. Otherwise, $v < 0$ and the right side of (4.2) is negative and congruent to $(v_2 - 1)(-v_1 - 1) \equiv -a \pmod{D}$. By varying q , we find that there are infinitely many elements of \mathcal{D} in the residue class $a \pmod{d}$. \square

Remarks. Equation (4.1) holds for any v coprime to D . Thus, if $a \equiv 2 \pmod{4}$ and either $(a + 1, D) = 1$ or $(a - 1, D) = 1$ then the residue class $a \pmod{d}$ contains infinitely many elements of \mathcal{D} ; take $v \equiv -a - 1 \pmod{d}, v > 0$ if $(a + 1, D) = 1$ and $v \equiv a - 1 \pmod{d}, v < -1$ if $(a - 1, D) = 1$. Thus, if d has at most two distinct prime factors, and (b) holds for some D composed only of the primes dividing d , then every residue class $a \pmod{d}$, with $2|a$ contains infinitely many elements of \mathcal{D} . Note that in this case, for all a with $2|a$, either $(a + 1, d) = 1$ or $(a - 1, d) = 1$. In particular this holds with d of the form 2^k , $2^k 3^\ell$, or $2^k 5^\ell$ with $k \geq 2$, since in each case (a) and (b) hold with $D = d$. Item (b) also holds with $d = D = 28$ (verified with PARI/GP). We do not know how to derive the same conclusion if d has 3 or more prime factors, e.g. $d = 60$.

REFERENCES

- [1] K. FORD, *Solutions of $\phi(n) = \phi(n + k)$ and $\sigma(n) = \sigma(n + k)$* , Internat. Math. Res. Notices (IMRN), to appear. <https://doi.org/10.1093/imrn/rnaa218>
- [2] E. FOUVRY AND M. WALDSCHMIDT, *Sur la représentation des entiers par les formes cyclotomiques de grand degré*, Bull. Soc. Math. France **148** (2020), 253–282.
- [3] A. GRANVILLE, D. M. KANE, D. KOUKOULOPOULOS AND R. J. LEMKE-OLIVER, *Best possible densities of Dickson m -tuples, as a consequence of Zhang-Maynard-Tao*. Analytic number theory, 133–144, Springer, Cham, 2015.
- [4] J. MAYNARD, *Small gaps between primes*. Ann. of Math. (2) **181** (2015), no. 1, 383–413.
- [5] ———, *Dense clusters of primes in subsets*, Compos. Math. **152** (2016), no. 7, 1517–1554.
- [6] D. H. J. POLYMATH, *Variants of the Selberg sieve, and bounded gaps between primes*, Res. Math. Sci. **1** (2014), Art. 12, 83 pp. Erratum: ibid. **2** (2015), Art. 15, 2 pp.
- [7] I. E. SHPARLINSKI, List of Open Problems, <https://web.maths.unsw.edu.au/~igorshparlinski/CharSumProjects.pdf>

- [8] Y. ZHANG, *Bounded gaps between primes*. *Annals of Mathematics* **179** (2014), 1121–1174.

DEPARTMENT OF MATHEMATICS, 1409 WEST GREEN STREET, UNIVERSITY OF ILLINOIS AT URBANA-
CHAMPAIGN, URBANA, IL 61801, USA

Email address: `ford@math.uiuc.edu`

STEKLOV INSTITUTE OF MATHEMATICS, 8 GUBKIN STREET, MOSCOW, 119991, RUSSIA

Email address: `konyagin@mi-ras.ru`