ZERO-FREE REGIONS FOR THE RIEMANN ZETA FUNCTION

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1. Introduction

The methods of Korobov [11] and Vinogradov [25] produce a zero-free region for the Riemann zeta function $\zeta(s)$ of the following strength: for some constant $c > 0$, there are no zeros of $\zeta(s)$ for $s = \beta + it$ with $|t|$ large and

$$1 - \beta \le \frac{c}{(\log |t|)^{2/3}(\log\log |t|)^{1/3}}. \tag{1.1}$$

The principal tool is an upper bound for $|\zeta(s)|$ near the line $\sigma = 1$. One form of this upper bound was given by Richert [17] as

$$|\zeta(\sigma + it)| \le A|t|^{B(1-\sigma)^{3/2}} \log^{2/3} |t| \qquad (|t| \ge 3, \tfrac{1}{2} \le \sigma \le 1) \tag{1.2}$$

with $B = 100$ and A and unspecified absolute constant. Subsequently, (1.2) was proved with smaller values of B, the best published value being 18.497 [12] (the author has a new result [7] that (1.2) holds with $B = 4.45$, $A = 76.2$).

Table 1 shows the historical progression of zero-free regions for $\zeta(s)$ prior to the work of Vinogradov and Korobov.

Zero-free region	Reference
$1 - \beta \le \dfrac{c}{\log \|t\|}$	de la Vallée Poussin [24], 1899
$1 - \beta \le \dfrac{c \log\log \|t\|}{\log \|t\|}$	Littlewood [13], 1922
$1 - \beta \le \dfrac{c}{(\log \|t\|)^{3/4+\varepsilon}}$	Chudakov [5], 1938

Table 1

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Recently, versions of (1.1) with explicit constants c have been given, valid for $|t|$ sufficiently large. Popov [15] showed that (1.1) holds with $c = 0.00006888$. Heath-Brown [8] proved (1.1) with $c \approx 0.0269B^{-2/3}$, and he noted (but did not give details) that the methods of [9] could be used to improve 0.0269 to about 0.0467. The main object of this note is to improve the constant c as a function of B.

Theorem 1. *If* (1.2) *holds with a certain constant* B*, then for large* $|t|$, $\zeta(\beta + it) \neq 0$ *for*

$$1 - \beta \leq \frac{0.05507B^{-2/3}}{(\log|t|)^{2/3}(\log\log|t|)^{1/3}}.$$

Taking $B = 4.45$ (from [7]) in Theorem 1 gives the zero-free region (1.1) with $c = \frac{1}{49.13}$. In addition, we prove a totally explicit zero-free region of type (1.1), with an explicit c and valid for all $|t| \geq 3$. This depends on both A and B in (1.2)), and may be used to give completely explicit bounds for prime counting functions (see e.g. [19], [20], [16]). Cheng [1] proved (1.2) with $A = 175$ and $B = 46$ and used this to deduce that (1.1) holds for all $|t| \geq 3$ with the constant $c = 1/990$. In turn, this result was used to show [2] that for all $x > 10$,

$$\left|\pi(x) - \operatorname{li}(x)\right| \leq 11.88x(\log x)^{3/5}\exp\{-\tfrac{1}{57}(\log x)^{3/5}(\log\log x)^{-1/5}\},$$

and that for $x \geq e^{e^{44.06}}$, there is a prime between x^3 and $(x+1)^3$ [3].

Theorem 2. *Suppose* (1.2) *holds,* $T_0 \geq e^{30000}$ *and* $\frac{\log T_0}{\log\log T_0} \geq \frac{1740}{B}$. *Suppose the zeros* $\beta + it$ *of* $\zeta(s)$ *with* $T_0 - 1 \leq t \leq T_0$ *all satisfy*

$$1 - \beta \geq \frac{M_1 B^{-2/3}}{(\log t)^{2/3}(\log\log t)^{1/3}}, \tag{1.3}$$

where

$$M_1 = \min\left(0.05507, \frac{0.1652}{2.9997 + \max_{t \geq T_0} X(t)/\log\log t}\right),$$

and

$$X(t) = 1.1585\log A + 0.859 + 0.2327\log\left(\frac{B}{\log\log t}\right) + \left(\frac{1.313}{B^{4/3}} - \frac{2.188}{B^{1/3}}\right)\left(\frac{\log\log t}{\log t}\right)^{\frac{1}{3}}.$$

Then (1.3) *is satisfied for all zeros with* $t \geq T_0$.

Since $\frac{0.1652}{2.9997} > 0.05507$, $M_1 = 0.05507$ when T_0 is sufficiently large. By classical zero density bounds (see e.g. Chapter 9 of [23]), for some positive δ, the number of zeros of $\zeta(s)$ is the rectangle $\frac{3}{4} \leq \Re s \leq 1, 0 < \Im s \leq T$ is $O(T^{1-\delta})$. Thus for most T_0, $\zeta(s)$ is zero free in the region $\frac{3}{4} \leq \Re s \leq 1, T_0 - 1 \leq \Im s \leq T_0$. Taking such T_0 which is sufficiently large, we see that Theorem 1 follows from Theorem 2.

To prove a totally explicit zero-free region of type (1.1) for $|t| \geq 3$, we make use of classical type (de la Valée Poussin type) zero-free regions for smaller $|t|$. These take the form

$$1 - \beta \leq \frac{c}{\log |t|} \qquad (|t| \geq 3). \tag{1.4}$$

Stechkin [21] proved (1.4) with $c = 1/9.646$ (he rounded this to $c = 9.65$ in his Theorem 2). Very tiny refinements were subsequently given by Rosser and Schoenfeld [20] and by Ramaré and Rumely [16]. With an explicit version of van der Corput's bound $|\zeta(1/2 + it)| \ll |t|^{1/6} \log |t|$ for $|t| \geq 3$, the methods of this paper produce a zero-free region

$$1 - \beta \leq \frac{1}{C_1(\log |t| + 6 \log\log |t|) + C_2}, \qquad (|t| \geq 3), \tag{1.5}$$

with $C_1 \approx 3.36$ and an explicit C_2. Better upper bounds are known for $|\zeta(1/2 + it)|$ for large t, the best being $O_\varepsilon(|t|^{89/570+\varepsilon})$ due to Huxley [10]. The implied constants are too large to improve the zero-free region, however. The zero-free region (1.5) also follows from Heath-Brown's methods with the same C_1 (and slightly larger C_3). In fact, the methods of this paper do not improve on Heath-Brown's methods when it comes to classical type zero-free regions for $\zeta(s)$ or zero-free regions for Dirichlet L-functions $L(s, \chi)$ when $|t|$ is small and the conductor of χ is large (e.g. those in [9]). Our methods do improve the Vinogradov-Korobov zero-free regions for $L(s, \chi)$ when the conductor of χ is fixed and $|t|$ becomes large.

It is known [14] that all zeros with $|\Im \rho| \leq 5.45 \times 10^8$ in fact lie on the critical line. Still, at $t = 5.45 \times 10^8$, $6 \log\log t \approx 0.895 \log t$, so improving greatly on Stechkin's region for all $|t| \geq 3$ with (1.5) is not possible. Still, we can make a modest improvement using the bound

$$|\zeta(1/2 + it)| \leq \min\left(6t^{1/4} + 57, 3t^{1/6} \log t\right) \qquad (t \geq 3). \tag{1.6}$$

proved by Cheng and Graham [4] [1].

[1][amended July, 2022] The proof in [4] contains an unfixable error, namely Lemma 3 is false (the best possible estimate in the Lemma was proved by Landau in 1927). Since my paper was published in 2002, I became aware of an older bound $|\zeta(1/2+it)| \leq 4(|t|/2\pi)^{1/4}$ for $|t| \geq 128\pi$ of R. S. Lehman, Proc. LMS (3) 20 (1970), p. 303–320, Lemma 2. This is better than the first bound in (1.6) and by itself leads to better numerical bounds in Theorem 4. The second bound in (1.6) has been improved recently by Trudgian and Hiary, although the published papers also use the erroneous Lemma 3 from [4]. Correcting the mistake leads to the bound $|\zeta(1/2+it)| \leq 0.77t^{1/6} \log |t|$, better than the 2nd claimed bound in (1.6). For details see the paper by D. Patel, *An Explicit Upper Bound for* $|\zeta(1 + it)|$, arXiv: 2009.00769, and also: Ghaith A. Hiary, Dhir Patel, and Andrew Yang, *An improved estimate for* $\zeta(1/2 + it)$, arXiv: 2207.02366

Theorem 3. *Let $T_0 = 5.45 \times 10^8$ and let*

$$J(t) = \min\left(\tfrac{1}{4}\log t + 1.8521, \tfrac{1}{6}\log t + \log\log t + \log 3\right). \tag{1.7}$$

Then $\zeta(\beta + it) \neq 0$ for $t \geq T_0$ and

$$1 - \beta \leq \frac{0.04962 - \frac{0.0196}{J(t)+1.15}}{J(t) + 0.685 + 0.155\log\log t}. \tag{1.8}$$

We note that $J(t)$ is an increasing function of t, and $(J(t) + 0.685 + 0.155\log\log t)/\log t$ is a decreasing function of t. Therefore, we conclude as a corollary that

Theorem 4. *We have $\zeta(\beta + it) \neq 0$ for $|t| \geq 3$ and*

$$1 - \beta \leq \frac{1}{8.463\log|t|}.$$

Further verification that the zeros of $\zeta(s)$ for some range of $t > 5.45 \times 10^8$ would give an improved constant in Theorem 4, as would an improvement in the bound for $|\zeta(1/2 + it)|$ in the vicinity of $t = T_0$.

We now return to the problem of producing a totally explicit zero-free regions of Korobov-Vinogradov type. Taking $B = 4.45$, $A = 76.2$ (from [7]), we find that

$$\begin{aligned} 1.1585\log A + 0.859 + 0.2327\log\left(\tfrac{B}{\log\log t}\right) + \left(\tfrac{1.313}{B^{4/3}} - \tfrac{2.188}{B^{1/3}}\right)\left(\tfrac{\log\log t}{\log t}\right)^{\frac{1}{3}} \\ \leq 6.22660 - 0.2327\log\log\log t - 1.1508\left(\frac{\log\log t}{\log t}\right)^{1/3} \\ \leq 5.6008. \end{aligned}$$

Thus

$$M_1 \geq \min\left(0.05507, \frac{0.1652}{2.9997 + \frac{5.6008}{\log\log T_0}}\right). \tag{1.9}$$

We take $T_0 = e^{54550}$, use Theorem 3 for $t \leq T_0 + 1$, and Theorem 2 plus (1.9) for larger t. This gives

Theorem 5. *The function $\zeta(\beta + it)$ is nonzero in the region*

$$1 - \beta \leq \frac{1}{57.54(\log|t|)^{2/3}(\log\log|t|)^{1/3}}, \quad |t| \geq 3.$$

2. The zero detector

Lemma 2.1. *Suppose f is the quotient of two entire functions of order $< k$, where k is a positive integer, and $f(0) \neq 0$. If z is neither a pole nor* *a zero of f, then*

$$\frac{f'(z)}{f(z)} = \sum_{|\rho|\le 2|z|} \frac{(z/\rho)^{k-1}}{z-\rho} m_\rho + O_f\left(|z|^{k-1}\right),$$

$$\left|\log|f(z)|\right| \le \left|\sum_{|\rho|\le 2|z|} \log\left|(1-z/\rho)e^{g(z/\rho)}\right|\right| + O_f\left(|z|^k\right),$$

where ρ runs over the zeros and poles of f (with multiplicity), $g(y) = y + \frac{1}{2}y^2 + \cdots + \frac{1}{k-1}y^{k-1}$, and m_ρ is either 1 (if ρ is a zero of f) or -1 (if ρ is a pole of f). The implied constants depend on f.

Proof. By theorems of Weierstrass and Hadamard ([22], Ch. VII, (2.13) and (10.1)),

$$f(z) = e^{f_1(z)} \prod_\rho \left[(1-z/\rho)e^{g(z/\rho)}\right]^{m_\rho},$$

where f_1 is a polynomial of degree $\le k$. Therefore, assuming that z is not a zero or pole of f, we have

$$\log|f(z)| = \Re f_1(z) + \sum_\rho m_\rho \left(\log|(1-z/\rho)e^{g(z/\rho)}|\right),$$

$$\frac{f'(z)}{f(z)} = f_1'(z) + \sum_\rho m_\rho \left(\frac{1}{z-\rho} + \frac{1}{\rho} + \frac{z}{\rho^2} + \cdots + \frac{z^{k-2}}{\rho^{k-1}}\right).$$

Now suppose $|\rho| > 2|z|$. We then have

$$\left|\frac{1}{z-\rho} + \frac{1}{\rho} + \frac{z}{\rho^2} + \cdots + \frac{z^{k-2}}{\rho^{k-1}}\right| = \left|\frac{(z/\rho)^{k-1}}{z-\rho}\right| \le 2\frac{|z|^{k-1}}{|\rho|^k}.$$

Since $\sum_\rho 1/|\rho|^k$ converges, the first part of the lemma follows. Similarly

$$\left|(1-z/\rho)e^{g(z/\rho)}\right| \le \exp\{\tfrac{2}{k}|\tfrac{z}{\rho}|^k\},$$

and the second part follows. □

The next lemma is the main "zero detector". Instead of integrating around a small circle centered at $z = z_0$ (as in [9], Lemma 3.2), we integrate over two vertical lines.

Lemma 2.2. *Suppose f is the quotient of two entire functions of finite order, and does not have a zero or a pole at $z = z_0$ nor at $z = 0$. Then, for all $\eta > 0$ except for a set of Lebesgue measure 0 (the exceptional set may* *depend on f and z_0), we have*

$$-\Re\frac{f'(z_0)}{f(z_0)} = \frac{\pi}{2\eta}\sum_{|\Re(z_0-\rho)|\le\eta} m_\rho \Re\cot\left(\frac{\pi(\rho-z_0)}{2\eta}\right) + \frac{1}{4\eta}\int_{-\infty}^{\infty}\frac{\log\left|f\left(z_0-\eta+\frac{2\eta iu}{\pi}\right)\right| - \log\left|f\left(z_0+\eta+\frac{2\eta iu}{\pi}\right)\right|}{\cosh^2 u}du,$$

where ρ runs over the zeros and poles of f (with multiplicity), and m_ρ is either 1 (if ρ is a zero of f) or -1 (if ρ is a pole of f).

Proof. We must exclude η for which the lines $\Re z = z_0 \pm \eta$ come "too close" to a zero or pole of f, since otherwise the above integral might not converge. By hypothesis, for some integer k, f is the quotient of two entire functions of order $< k$. We say a positive real number η is "good" if there is a positive number δ such that for every zero/pole ρ of f, $|\Re(\rho - z_0) \pm \eta| \ge \delta|\rho|^{-k}$. The number δ may depend on η. Since $\sum_\rho |\rho|^{-k}$ converges, the set of η for which $|\Re(\rho - z_0) \pm \eta| \le \delta|\rho|^{-k}$ has measure $O(\delta)$ (here and throughout this proof, implied constants depend on f and z_0). Taking $\delta \to 0$ shows that the measure of "bad" η is 0.

Suppose now that η is "good" with an associated number δ. We may assume that $0 < \delta \le 1$. Let T be a large real number such that $T \ge \eta$, $T \ge 2|z_0|$ and for all zeros/poles ρ of f, $|\Im(\rho - z_0) \pm T| \ge |\rho|^{-k}$. Since $\sum_\rho |\rho|^{-k}$ converges, the set of "bad" T has measure $O(1)$. Consider the contour $C = C_1 \cup C_2 \cup C_3 \cup C_4$, where the C_j are the line segments connecting the points $\eta - iT, \eta + iT, -\eta + iT, -\eta - iT, \eta - iT$, respectively. Let

$$I = I_1 + I_2 + I_3 + I_4, \quad I_j = \int_{C_j} \frac{f'(z+z_0)}{f(z+z_0)} h(z)\, dz,$$

where

$$h(z) = \frac{\pi}{2\eta}\cot\left(\frac{\pi z}{2\eta}\right).$$

By Cauchy's Residue Theorem,

$$I = \frac{f'(z_0)}{f(z_0)} + \sum_{\substack{|\Re(\rho-z_0)|\le\eta \\ |\Im(\rho-z_0)|\le T}} m_\rho h(\rho - z_0). \tag{2.1}$$

There is a holomorphic branch of $\log f(z + z_0)$ on C^*, the contour C cut at the point η. Applying integration by parts, and noting that $h(\eta) = 0$, we have

$$(2.2)\qquad \begin{aligned} I &= \lim_{\varepsilon\to 0^+} [h(z)\log f(z+z_0)]_{\eta+i\epsilon}^{\eta-i\epsilon} - \frac{1}{2\pi i}\int_{C^*} h'(z)\log f(z+z_0)\,dz \\ &= -(J_1+J_2+J_3+J_4), \qquad J_j = \frac{1}{2\pi i}\int_{C_j} h'(z)\log f(z+z_0)\,dz. \end{aligned}$$

The number of zeros/poles ρ with $|\rho| \le x$ is $O(x^k)$, and $|\rho| \gg 1$ for every ρ. By our assumptions about T, when $z \in C$ we have $|z+z_0| \ll T$. Therefore, by Lemma 2.1 and our assumption about η,

$$\begin{aligned} \left|\frac{f'(z+z_0)}{f(z+z_0)}\right| &\ll T^{k-1} + \sum_{|\rho|\le 2|z|} \frac{|(z+z_0)/\rho|^{k-1}}{|z+z_0-\rho|} \\ &\ll T^{k-1} + \frac{T^{k-1}}{|\rho|^{k-1}}\frac{|\rho|^k}{\delta} \ll \delta^{-1}T^k. \end{aligned}$$

Likewise, using the second part of Lemma 2.1,

$$|\log|f(z+z_0)|| = O(T^{2k-1} + T^k\log(T\delta^{-1}))$$

for $z \in C$. Thus, there is a branch of $\log f(z+z_0)$ with

$$|\log f(z+z_0)| \ll T^{2k}\delta^{-1}.$$

This is important to the estimation of J_2 and J_4. Since

$$h'(z) = -\frac{\pi^2}{4\eta^2}\csc^2\left(\frac{\pi z}{2\eta}\right),$$

we have $|h'(\eta \pm iT)| \ll \eta^{-2}e^{-\pi T/(2\eta)}$. Therefore, $|J_2|+|J_4| \to 0$ as $T \to \infty$. Parameterizing the line segments C_1 and C_3 with $z = \pm\eta + \frac{2\eta i u}{\pi}$ and taking real parts gives

$$\Re(J_1+J_3) = \frac{1}{4\eta}\int_{-\frac{\pi T}{2\eta}}^{\frac{\pi T}{2\eta}} \frac{\log\left|f\left(z_0-\eta+\frac{2\eta i u}{\pi}\right)\right| - \log\left|f\left(z_0+\eta+\frac{2\eta i u}{\pi}\right)\right|}{\cosh^2 u}du.$$

Recalling (2.1) and (2.2), this proves the lemma upon letting $T \to \infty$. □

3. Bounds for $\zeta(s)$

Lemma 3.1. *Suppose $1 < \sigma \le 1.06$ and t is real. Then*

$$\frac{1}{\zeta(\sigma)} \le |\zeta(\sigma+it)| \le \zeta(\sigma) \le 0.6 + \frac{1}{\sigma-1}$$

and, for all $\sigma > 1$ and real t we have

$$\left|-\frac{\zeta'}{\zeta}(\sigma+it)\right| < \frac{1}{\sigma-1}.$$

Proof. For the first line of inequalities, we start with

$$|\zeta(\sigma+it)| \le \sum_{n=1}^\infty n^{-\sigma} = \zeta(\sigma)$$

and similarly

$$|\zeta(\sigma+it)|^{-1} = \left|\sum_{n=1}^\infty \mu(n) n^{-\sigma-it}\right| \le \sum_{n=1}^\infty n^{-\sigma} = \zeta(\sigma).$$

Next, since $x^{-\sigma}$ is convex and $e^{-y} \le 1-y+\frac12 y^2$ for $0\le y\le 1$, we have

$$\begin{aligned}\zeta(\sigma) &\le 1 + \int_{3/2}^\infty \frac{du}{u^\sigma} = 1 + \frac{(3/2)^{-(\sigma-1)}}{\sigma-1}\\ &\le 1 + \frac{1}{\sigma-1} - \log(1.5) + \tfrac12(\sigma-1)\log^2(1.5)\\ &\le 0.6 + \frac{1}{\sigma-1}.\end{aligned}$$

In fact, near $\sigma=1$ we have $\zeta(\sigma) = \frac{1}{\sigma-1} + \gamma + O(\sigma-1)$, where $\gamma = 0.5772\cdots$ is the Euler-Mascheroni constant (see e.g. [23], (2.1.16)). The last inequality in the lemma follows from $|-\frac{\zeta'}{\zeta}(\sigma+it)| \le -\frac{\zeta'}{\zeta}(\sigma)$ and

$$-\zeta'(\sigma) = \sum_{n=1}^\infty \left(\sum_{m\ge n+1} m^{-\sigma}\right)\log\left(\frac{n+1}{n}\right) < \sum_{n=1}^\infty \frac{n^{1-\sigma}}{\sigma-1}\frac{1}{n} = \frac{\zeta(\sigma)}{\sigma-1}.$$

□

Lemma 3.2. *For real u,*

$$\left|\frac{\zeta'(-\frac12+iu)}{\zeta(-\frac12+iu)}\right| \le 4.62 + \frac12\log(1+u^2/9).$$

Proof. (Corrected July, 2022). By the functional equation for $\zeta(s)$ (cf. [6], Ch. 12, (8)–(10)),

$$-\frac{\zeta'(w)}{\zeta(w)} = \frac{\zeta'(1-w)}{\zeta(1-w)} - \log\pi - \gamma - \sum_{n=1}^\infty\left(\frac{1}{w+2n} + \frac{1}{1-w+2n} - \frac1n\right) + \frac{1}{w(w-1)}.$$

Now set $w = -\frac12 + iu$. A short numerical calculation shows that

$$\max_u \left|-\log\pi - \gamma + \frac{1}{(-1/2+iu)(-3/2+iu)}\right| \le 1.877$$

and that

$$|\zeta'(1-w)/\zeta(w)| \le -\zeta'(3/2)/\zeta(3/2) \le 1.506.$$

Therefore,

$$\begin{aligned}\left|\frac{\zeta'(w)}{\zeta(w)}\right| &\le 3.383 + \sum_{n=1}^{\infty}\left|\frac{u^2+n-3/4+2iu}{n(4n^2+2n-3/4+u^2+2iu)}\right| \\ &\le 4.383 + \sum_{n=1}^{\infty}\frac{n-3/4}{n(4n^2+2n-3/4)} + \sum_{n=2}^{\infty}\frac{|u^2+2iu|}{n(4n^2+u^2)} \\ &\le 4.542 + |u|\sqrt{u^2+4}\int_{3/2}^{\infty}\frac{dx}{x(4x^2+u^2)} \\ &= 4.542 + \frac{\sqrt{u^2+4}}{2|u|}\log(1+u^2/9) \\ &\le 4.62 + \tfrac{1}{2}\log(1+u^2/9),\end{aligned}$$

the last line following from another numerical calculation. □

Lemma 3.3. *We have*

$$\sum_{\rho}\frac{1}{|\rho|^2} \le 0.0463,$$

where the sum is over all of the non-trivial zeros of $\zeta(s)$.

Proof. By ([6], Ch. 9, (10) an (11)), we have

$$\sum_{\rho}\frac{\Re\rho}{|\rho|^2} = 1 + \tfrac{1}{2}\gamma - \tfrac{1}{2}\log(4\pi).$$

If $\zeta(\rho) = 0$ then $\zeta(1-\rho) = 0$, and the minimum of $|\Im\rho|$ is > 14.1. Thus

$$\begin{aligned}\sum_{\rho}\frac{1}{|\rho|^2} &= \sum_{\rho}\left(\frac{\Re\rho}{|\rho|^2} + \frac{\Re\rho}{|1-\rho|^2}\right) \\ &\le \left(1+\sqrt{1+1/14.1^2}\right)\sum_{\rho}\frac{\Re\rho}{|\rho|^2} \\ &\le 0.0463.\end{aligned}$$

□

Lemma 3.4. *Let us fix* $\sigma \in [\frac{1}{2}, 1)$, *and suppose for all* $t \ge 3$ *we have*

$$|\zeta(\sigma+iy)| \le Xt^Y(\log t)^Z \qquad (1 \le |y| \le t), \tag{3.1}$$

where X, Y *and* Z *are positive constants with* $Y + Z \ge 0.1$. *If* $0 < a \le \frac{1}{2}$, $t \ge 100$ *and* $\frac{1}{2} \le \sigma \le 1 - 1/t$, *then*

$$\int_{-\infty}^{\infty}\frac{\log|\zeta(\sigma+it+iau)|}{\cosh^2 u}\,du \le 2(\log X + Y\log t + Z\log\log t).$$

Proof. First, there is no difficulty if $\zeta(\sigma+it+iau)=0$ for some points along the path of integration. Since all zeros have finite order, the integral in the lemma always converges. When $-\frac{2t}{a}\le u\le \frac{-t-1}{a}$, (3.1) gives $|\zeta(\sigma+it+iau)|\le Xt^Y(\log t)^Z$. For $\frac{-t-1}{a}\le u\le\frac{-t+3}{a}$, we use the identity ([23], (2.1.4))

$$\zeta(s)=\frac{1}{s-1}+\frac{1}{2}+s\int_1^\infty \frac{\lfloor x\rfloor - x+1/2}{x^{s+1}}\,dx.$$

Writing $s=\sigma+it+iau$, it follows that $|s-1|\ge 1/t$ and $|s|\le\sqrt{10}$ and thus $\log|\zeta(s)|\le\log(t+4)$ for this range of u. For $u\ge\frac{3-t}{a}$, we use the inequalities $\log(1+x)\le x$ and $\log(1+x)\le x-\frac12 x^2+\frac13 x^3$, both valid for all $x>-1$. Then

$$\begin{aligned}\log|\zeta(\sigma+it+iau)| &\le \log X+Y\log(t+au)+Z\log\log(t+au)\\ &\le \log(Xt^Y(\log t)^Z)+\left(Y+\frac{Z}{\log t}\right)\left(\frac{au}{t}-\frac{(au)^2}{2t^2}+\frac{(au)^3}{3t^3}\right).\end{aligned}$$

Similarly, using $\log(1+x)\le x$, for $u\le -\frac{2t}{a}$

$$\log|\zeta(\sigma+it+iau)|\le \log(Xt^Y(\log t)^Z)+\left(Y+\frac{Z}{\log t}\right)\left(\frac{-au-2t}{t}\right).$$

Combining these estimates together with $\int_{-\infty}^\infty(\cosh u)^{-2}\,du=2$ yields

$$\int_{-\infty}^\infty \frac{\log|\zeta(\sigma+it+iau)|}{\cosh^2 u}\,du\le 2(\log X+Y\log t+Z\log\log t)+E,$$

where

$$\begin{aligned}E=&\frac{4\log(t+4)}{a\cosh^2\left(\frac{3-t}{a}\right)}\\ &+\left(Y+\frac{Z}{\log t}\right)\left(\int_{-\infty}^{-\frac{2t}{a}}\frac{-au-2t}{t\cosh^2 u}\,du+\int_{\frac{3-t}{a}}^\infty \frac{\frac{au}{t}-\frac{(au)^2}{2t^2}+\frac{(au)^3}{3t^3}}{\cosh^2 u}\,du\right).\end{aligned}$$

Now $\frac14 e^{2|u|}\le\cosh^2 u\le e^{2|u|}$, $a\le\frac12$ and $t\ge 100$. Hence

$$ae^{(t-6)/a}\ge 2e^{2t-12}.$$

Therefore

$$\begin{aligned} E &\le \frac{16\log(t+4)}{ae^{2(t-3)/a}} + \left(Y + \frac{Z}{\log t}\right)\left(\tfrac{4a}{t}e^{-4t/a}\int_0^\infty ve^{-2v}\,dv \right. \\ &\qquad \left. + \int_{-\infty}^{\infty} \frac{\frac{au}{t} - \frac{(au)^2}{2t^2} + \frac{(au)^3}{3t^3}}{\cosh^2 u}\,du + \int_{\frac{t-3}{a}}^{\infty} \frac{\frac{au}{t} + \frac{(au)^2}{2t^2} + \frac{(au)^3}{3t^3}}{\frac14 e^{2u}}\,du\right) \\ &\le \frac{32\log(t+4)}{e^{t/a+2t-12}} + \left(Y + \frac{Z}{\log t}\right)\left(\frac{e^{-4t/a}}{t} - \frac{\pi^2 a^2}{12t^2} + \frac{8a^3}{t^3}\int_{2t-6}^{\infty} u^3 e^{-2u}\,du\right) \\ &\le e^{-t/a} + \left(Y + \frac{Z}{\log t}\right)\left(e^{-4t/a} - \frac{\pi^2}{12}\frac{a^2}{t^2} + 48a^3e^{-4t+12}\right) \\ &\le e^{-t/a} - \frac{0.1}{\log t}\frac{a^2}{2t^2} \\ &\le 0. \end{aligned}$$

□

4. Detecting zeros of $\zeta(s)$

From now on, ρ will denote a zero of $\zeta(s)$ and in summations over the zeros, each zero is counted according to its multiplicity. Since $\zeta(s) = \overline{\zeta(\overline{s})}$, when proving zero-free regions we restrict our attention to the upper half plane.

Lemma 4.1. *Suppose* (1.2) *holds. Let* $s = \sigma + it$, $\eta > 0$, $\sigma - \eta \ge 1/2$, $1 \le \sigma \le 1 + \eta$ *and* $t \ge 100$. *If* S *is any subset of* $\{z : \sigma - \eta \le \Re z \le 1\}$, *then*

$$\begin{aligned} -\Re\frac{\zeta'(s)}{\zeta(s)} \le &- \sum_{\rho\in S,\zeta(\rho)=0} \Re\frac{\pi}{2\eta}\cot\left(\frac{\pi(s-\rho)}{2\eta}\right) \\ &+ \frac{1}{2\eta}\left(\frac{2}{3}\log\log t + B(1-\sigma+\eta)^{3/2}\log t + \log A\right) \\ &- \frac{1}{4\eta}\int_{-\infty}^{\infty}\frac{\log|\zeta(s+\eta+2\eta iu/\pi)|}{\cosh^2 u}\,du. \end{aligned}$$

Proof. We apply Lemma 2.2 with $f = \zeta$ and $z_0 = s$, noting that $\zeta(0) \ne 0$, all zeros have real part < 1 and that $\Re\cot z \ge 0$ for $0 \le \Re z \le \frac{\pi}{2}$. Thus the right side in the conclusion of Lemma 2.2 is increased if we omit from the sum any subset of the zeros. Then we apply (1.2) and Lemma 3.4 (with $X = A$, $Y = B(1-\sigma+\eta)^{3/2}$, $Z = 2/3$, $a = 2\eta/\pi$) to the integral over the line $\Re z = \sigma - \eta$. Note also that the integral on the right side in Lemma 4.1 always converges by Lemma 3.1. Therefore, if η is “bad” with respect to Lemma 2.2, we can apply the above argument with a sequence of numbers η' tending to η from above. □

We next require an upper bound on the number of zeros close to a point $1+it$. Here $N(t,R)$ denotes the number of zeros ρ with $|1+it-\rho| \le R$.

Lemma 4.2. *Assume* (1.2) *holds with* $A > 1$ *and* $B > 0$. *Then, for* $0 < R \le 1/4$, $t \ge 100$,

$$N(t,R) \le 1.3478 R^{3/2} B \log t + 0.49 + \frac{\log A - \log R + \frac{2}{3}\log\log t}{1.879}.$$

Proof. Apply Lemma 4.1 with $s = 1 + 0.6421R + it$, $\eta = 2.5R$ (so that $\sigma - \eta \ge \frac{1}{2}$) and $S = \{z : |1+it-z| \le R, \Re z \le 1\}$. These parameters were chosen to minimize the first term on the right side of the inequality in the lemma. By Lemma 3.1, if v is real then

$$\begin{aligned} &\left|\frac{\zeta'}{\zeta}(s+\eta+iv)\right| \le \frac{1}{3.1421R}, \\ &|\zeta(s+\eta+iv)|^{-1} \le \zeta(1+3.1421R) \le 0.6 + \frac{1}{3.1421R}. \end{aligned} \tag{4.1}$$

Next, in the region $U = \{z : \Re z \ge 0.6421, |z - 0.6421| \le 1\}$, we prove

$$\Re \frac{\pi}{5} \cot\left(\frac{\pi z}{5}\right) \ge 0.3758. \tag{4.2}$$

By the maximum modulus principle, it suffices to prove (4.2) on the boundary of U. Using

$$\Re \cot(x+iy) = \frac{2\sin(2x)}{e^{2y} + e^{-2y} - 2\cos(2x)},$$

the minimum of $\Re\cot(x+iy)$ on the vertical segment $x = 0.6421\pi/5$, $|y| \le \pi/5$ occurs at the endpoints. On the semicircular part of the boundary of U, we verified (4.2) by a short computation using the computer algebra package Maple. In particular, the relative minima on the boundary of U occur at $z = 1.6421$ and $z = 0.6421 \pm i$. Therefore, by (4.1), (4.2) and Lemma 4.1,

$$\begin{aligned} -\frac{1}{3.1421R} \le -0.3758\frac{N(t,R)}{R} + \frac{1}{5R}\Bigg(&\frac{2}{3}\log\log t + \\ &(1.8579R)^{3/2} B \log t + \log A + \log\left(0.6 + \tfrac{1}{3.1421R}\right)\Bigg). \end{aligned}$$

Since $\log(0.6 + \frac{1}{3.1421R}) \le -\log(3.1421R) + 1.88526R \le -\log R - 0.6735$, the lemma follows. $\square$

Remark. A qualitatively similar result may also be proved, in a similar way, from Lemma 2 of [8], or from Landau's lemma (§3.9 of [23]).

Lemma 4.3. *Suppose $t \geq 10000$, $0 < v \leq 1/4$, and (1.2) holds with $A > 1$, $B > 0$. Then*

$$\sum_{|1+it-\rho|\geq v} \frac{1}{|1+it-\rho|^2} \leq (6.132 + 5.392B(v^{-1/2}-2))\log t + 13.5$$

$$- 8.5\log A + 4\log\log t + \frac{\frac{\log A - \log v + \frac{2}{3}\log\log t}{1.879} + 0.224 - N(t,v)}{v^2}.$$

Proof. Divide the zeros with $|1+it-\rho| \geq v$ into three sets:

$$Z_1 = \{\rho : |\Im\rho - t| \geq 1\},$$
$$Z_2 = \{\rho \notin Z_1 : |1+it-\rho| \geq \tfrac{1}{4} \text{ and } |it-\rho| \geq \tfrac{1}{4}\},$$
$$Z_3 = \{\rho : \rho \notin Z_2, \rho \notin Z_1 \text{ and } |1+it-\rho| \geq v\}.$$

For $i = 1,2,3$, let S_i be the sum over $\rho \in Z_i$ of $|1+it-\rho|^{-2}$. By Theorem 19 of [18], the number, $N(T)$, of nontrivial zeros of $\zeta(s)$ with imaginary part in $[0,T]$ satisfies

$$N(T) = \frac{T}{2\pi}\log\frac{T}{2\pi} - \frac{T}{2\pi} + \frac{7}{8} + Q(T), \tag{4.3}$$

where

$$|Q(T)| \leq 0.137\log T + 0.443\log\log T + 1.588 \qquad (T \geq 2).$$

Since there are no zeros ρ with $|\Im\rho| \leq 14$,

$$S_1 \leq \int_{t+1}^{\infty} \frac{dN(u)}{(u-t)^2} + \int_{14}^{t-1} \frac{dN(u)}{(t-u)^2} + \int_{14}^{\infty} \frac{dN(u)}{(u+t)^2} = I_1 + I_2 + I_3.$$

Since $dN(u) = \frac{1}{2\pi}\log\frac{u}{2\pi} + dQ(u)$, $\log(t+x) \leq \log t + \frac{x}{t}$ and $\log\log(t+x) \leq \log\log t + \frac{x}{t\log t}$, we have

$$\begin{aligned} I_1 &\leq \frac{1}{2\pi}\int_1^{\infty} \frac{\log(t+x) - \log 2\pi}{x^2}\,dx + |Q(t+1)| + 2\int_1^{\infty} \frac{|Q(t+x)|}{x^3}\,dx \\ &= \frac{\left(1+\frac{1}{t}\right)\log(1+t) - \log(2\pi)}{2\pi} + |Q(t+1)| + 2\int_1^{\infty}\frac{|Q(t+x)|}{x^3}\,dx \\ &\leq 0.4332\log t + 0.886\log\log t + 2.884 + 2\int_1^{\infty}\frac{0.1851x/t}{x^3}\,dx \\ &\leq 0.4332\log t + 0.886\log\log t + 2.885. \end{aligned}$$

Similarly, noting that $Q(14) \geq 0$, we get

$$\begin{aligned} I_2 &\leq \frac{1}{2\pi}\log\left(\frac{t}{2\pi}\right) + 2\max_{14\leq u\leq t-1}|Q(u)| \\ &\leq 0.4332\log t + 0.886\log\log t + 2.884 \end{aligned}$$

and

$$I_3 \le \frac{1}{2\pi}\int_{14}^{\infty} \frac{\log\left(\frac{u+t}{2\pi}\right)}{(u+t)^2}\,du + 2\int_{14}^{\infty}\frac{|Q(u)|}{(u+t)^3}\,du \le 0.00014.$$

Thus

$$S_1 \le 0.8664\log t + 1.772\log\log t + 5.77. \tag{4.4}$$

Next let $N_2 = |Z_2|$ and $N_3 = |Z_3|$. By (4.3),

$$\begin{aligned} N_2 + N_3 &= N(t+1) - N(t-1) - N(t,v) \\ &\le 0.59231\log t + 0.886\log\log t + 2.591 - N(t,v). \end{aligned} \tag{4.5}$$

In the sum S_2, each zero on the critical line contributes ≤ 4 and each pair of zeros $\rho = \beta + i\gamma$, $\rho' = 1-\beta+i\gamma$ with $\beta > 1/2$ contributes at most $4^2 + (4/3)^2$ to the sum. Therefore,

$$S_2 \le \frac{80N_2}{9}.$$

For S_3, $N(t,1/4)$ of the zeros contribute at most $(4/3)^2$ each, since $N_3 + N(t,v) = 2N(t,1/4)$. By partial summation,

$$\begin{aligned} S_3 &\le \frac{16N(t,1/4)}{9} + \int_v^{1/4}\frac{dN(t,u)}{u^2} \\ &= \frac{160}{9}N(t,1/4) - \frac{N(t,v)}{v^2} + 2\int_v^{1/4}\frac{N(t,u)}{u^3}\,du \\ &= \frac{80N_3}{9} + \left(\frac{80}{9} - \frac{1}{v^2}\right)N(t,v) + 2\int_v^{1/4}\frac{N(t,u)}{u^3}\,du. \end{aligned}$$

By Lemma 4.2,

$$\begin{aligned} 2\int_v^{1/4}\frac{N(t,u)}{u^3}\,du &\le \left(\frac{\log A + \frac{2}{3}\log\log t}{1.879} + 0.49\right)\left(v^{-2} - 16\right) \\ &\quad + 5.3912B\left(v^{-1/2} - 2\right)\log t + \frac{1}{1.879}\left(8 - 16\log 4 - \frac{1 + 2\log v}{2v^2}\right). \end{aligned}$$

Therefore, using (4.5), we obtain

$$\begin{aligned} S_2 + S_3 &\le (5.2650 + 5.3912B(v^{-1/2} - 2))\log t + 2.2\log\log t - 8.5\log A \\ &\quad + 7.65 + \frac{1}{v^2}\left(\frac{\log A - \log v + \frac{2}{3}\log\log t}{1.879} + 0.224\right) - \frac{N(t,v)}{v^2}. \end{aligned}$$

Combining this with (4.4) gives the lemma. □

Lemma 4.4. *Suppose that* $\Re z \ge 0$ *and* $|z| \le \pi/2$. *Then*

$$\Re\left(\cot z - \frac{1}{z} + \frac{4z}{\pi^2}\right) \ge 0.$$

Proof. By the maximum modulus principle it suffices to prove the inequality on the boundary of the region. On the vertical segment $z = iy$, $-\pi/2 \le y \le \pi/2$, the left side is zero. When $|z| = \pi/2$, $z = x + iy$ and $x \ge 0$, the left side is

$$\frac{2\sin(2x)}{e^{2y}+e^{-2y}-2\cos(2x)} - \frac{x}{x^2+y^2} + \frac{4x}{\pi^2} = \frac{2\sin(2x)}{e^{2y}+e^{-2y}-2\cos(2x)} \ge 0.$$

This proves the lemma. □

The next two lemmas are related to Heath-Brown's method for detecting zeros from [9]. These give bounds for a "mollified" sum, similar to Lemmas 5.1 and 5.2 of [9].

Lemma 4.5. *Suppose f is a non-negative real function which has continuous derivative on $(0,\infty)$. Suppose the Laplace transform*

$$F(z) = \int_0^\infty f(y)e^{-zy}\,dy$$

of f is absolutely convergent for $\Re z > 0$. Let $F_0(z) = F(z) - f(0)/z$ and suppose

$$|F_0(z)| \le \frac{D}{|z|^2} \qquad (\Re z \ge 0, |z| \ge \eta), \tag{4.6}$$

where $0 < \eta \le \frac{3}{2}$. If $\Re s > 1$ and $\Im s \ge 0$, then

$$\begin{aligned} K(s) &:= \sum_{n=1}^{\infty} \Lambda(n)n^{-s}f(\log n) \\ &= -f(0)\frac{\zeta'(s)}{\zeta(s)} - \sum_{\rho} F_0(s-\rho) + F_0(s-1) + E, \end{aligned}$$

where $|E| \le D(1.72 + \frac{1}{3}\log(1+\Im s))$.

Proof. We follow the proof of Lemma 5.1 of [9]. Suppose $s = \sigma + it$ and $1 < \alpha < \sigma$. Define

$$I = \frac{1}{2\pi i}\int_{\alpha-i\infty}^{\alpha+i\infty} -\frac{\zeta'(w)}{\zeta(w)}F_0(s-w)\,dw.$$

Since $-\zeta'(w)/\zeta(w) = \sum_n \Lambda(n)n^{-w}$, the sum converging uniformly on $\Re w = \alpha$, we may integrate term by term. Thus $I = \sum_n \Lambda(n)J_n$, where

$$J_n = \frac{1}{2\pi i}\int_{\alpha-i\infty}^{\alpha+i\infty} n^{-w}F_0(s-w)\,dw = \frac{n^{-s}}{2\pi i}\int_{\sigma-\alpha-i\infty}^{\sigma-\alpha+i\infty} n^{u}F_0(u)\,du.$$

The integral on the right converges absolutely by (4.6). Since

$$F_0(z) = \frac{1}{z}\int_0^\infty e^{-zy}f'(y)\,dy,$$

we have

$$J_n = \frac{n^{-s}}{2\pi i}\int_0^\infty f'(y)\int_{\sigma-\alpha-i\infty}^{\sigma-\alpha+i\infty} \frac{(ne^{-y})^u}{u}\,du\,dy$$
$$= n^{-s}\int_0^{\log n} f'(y)\,dy = n^{-s}\left(f(\log n) - f(0)\right).$$

Thus

$$I = K(s) + f(0)\frac{\zeta'(s)}{\zeta(s)}. \tag{4.7}$$

Moving the line of integration to $\Re w = -1/2$, we have

$$I = \frac{1}{2\pi i}\int_{-1/2-i\infty}^{-1/2+i\infty} -\frac{\zeta'(w)}{\zeta(w)}F_0(s-w)\,dw - \sum_\rho F_0(s-\rho) + F_0(s-1). \tag{4.8}$$

By (4.6) and Lemma 3.2, the integral in (4.8) is $\le \frac{D}{2\pi}I'$, where

$$I' \le \int_{-\infty}^\infty \frac{4.62 + \frac12\log(1+u^2/9)}{9/4+(u-t)^2}\,du$$
$$= 3.08\pi + \frac13\int_{-\infty}^\infty \frac{\log(1+(t/3+v/2)^2)}{1+v^2}\,dv$$
$$\le 3.08\pi + \frac13\int_{-\infty}^\infty \frac{\log(1+t^2)+\log(1+v^2)}{1+v^2}\,dv$$
$$\le 10.8 + \frac{2\pi\log(1+t)}{3}.$$

The lemma now follows from (4.7) and (4.8). □

Remarks. Examples of functions f satisfying the conditions of Lemma 4.5 are those with compact support (say $[0,x_0]$) and with f'' continuous and bounded on $(0,x_0)$. These are the functions considered in [9]. To see that (4.6) holds, apply integration by parts twice, noting that $f(x_0) = f'(x_0) = 0$. This gives

$$F_0(z) = z^{-2}\left(f'(0^+) + \int_0^{x_0} e^{-zt}f''(t)\,dt\right).$$

Lemma 4.6. *Suppose* $0 < \eta \le \frac12$ *and* (1.2) *holds with* $A > 1$, $B > 0$. *Let* f *have compact support and satisfy* (4.6). *Suppose* $s = 1+it$ *with* $t \ge 1000$.

Then

$$\Re K(s) \le -\sum_{|1+it-\rho|\le\eta} \Re\left\{F(s-\rho) + f(0)\left(\frac{\pi}{2\eta}\cot\left(\frac{\pi(s-\rho)}{2\eta}\right) - \frac{1}{s-\rho}\right)\right\}$$
$$+ \frac{f(0)}{2\eta}\left[\frac{2\log\log t}{3} + B\eta^{3/2}\log t + \log A - \frac{1}{2}\int_{-\infty}^{\infty} \frac{\log|\zeta(s+\eta+\frac{2\eta ui}{\pi})|}{\cosh^2 u}\,du\right]$$
$$+ D\left(1.8 + \frac{\log t}{3} + \sum_{|1+it-\rho|\ge\eta} \frac{1}{|1+it-\rho|^2}\right).$$

In addition,

$$K(1) \le F(0) + 1.8D.$$

Proof. Suppose that $\sigma > 1$. By Lemma 4.5,

$$K(\sigma) \le -f(0)\frac{\zeta'(\sigma)}{\zeta(\sigma)} + F_0(\sigma-1) + 1.72D + D\sum_{\rho} \frac{1}{|1-\rho|^2}.$$

Since $\zeta(\rho) = 0$ implies $\zeta(1-\rho) = 0$, we may replace $|1-\rho|^2$ by $|\rho|^2$ in the last sum. Using Lemmas 3.1 and 3.3, we obtain

$$\begin{aligned} K(\sigma) &\le \frac{f(0)}{\sigma-1} + F_0(\sigma-1) + 1.8D \\ &= F(\sigma-1) + 1.8D. \end{aligned} \tag{4.9}$$

When $t \ge 1000$ and $s = \sigma + it$, $\Re F_0(s-1) \le |F_0(s-1)| \le Dt^{-1} \le 0.001D$. Also by (4.6),

$$\sum_{|1+it-\rho|>\eta} |F_0(s-\rho)| \le D\sum_{|1+it-\rho|>\eta} \frac{1}{|1+it-\rho|^2}.$$

Therefore, combining Lemma 4.1 (with $S = \{z : \Re z \le 1,\ |1+it-z| \le \eta\}$) and Lemma 4.5 gives

(4.10)

$$\Re K(s) \le -\sum_{|1+it-\rho|\le\eta} \Re\left\{F(s-\rho) + f(0)\left(\frac{\pi}{2\eta}\cot\left(\frac{\pi(s-\rho)}{2\eta}\right) - \frac{1}{s-\rho}\right)\right\}$$
$$+ \frac{f(0)}{2\eta}\left[\frac{2}{3}\log\log t + B\eta^{3/2}\log t + \log A - \frac{1}{2}\int_{-\infty}^{\infty} \frac{\log|\zeta(s+\eta+\frac{2\eta ui}{\pi})|}{\cosh^2 u}\,du\right]$$
$$+ D\left(1.8 + \frac{\log t}{3} + \sum_{|1+it-\rho|>\eta} \frac{1}{|1+it-\rho|^2}\right).$$

Since f has compact support, $K(s)$ and $F(s)$ are both entire functions. Also, on the right side of (4.10), $|\log|\zeta(\alpha+i\beta)|| \le |\log\zeta(\alpha)|$ when $\alpha > 1$

(by Lemma 3.1). Thus we may let $\sigma \to 1^+$ in (4.9) and (4.10), and this proves the lemma. $\square$

5. A trigonometric inequality

We use a trigonometric inequality that is very similar to what is used in standard treatments. For any real numbers a_1, a_2 we have

$$\sum_{j=0}^{4} b_j \cos(j\theta) = 8(\cos\theta + a_1)^2(\cos\theta + a_2)^2 \ge 0 \quad (\theta \in \mathbb{R}), \tag{5.1}$$

where

$$\begin{aligned} &b_4 = 1, \quad b_3 = 4(a_1 + a_2), \quad b_2 = 4(1 + a_1^2 + a_2^2 + 4a_1a_2), \\ &b_1 = (a_1 + a_2)(12 + 16a_1a_2), \quad b_0 = b_2 - 1 + 8(a_1a_2)^2. \end{aligned} \tag{5.2}$$

Lemma 5.1. *Suppose a_1, a_2 are real numbers and define $b_0, \ldots, b_4$ by (5.2). Suppose that $\eta > 0$ and t_1, t_2 are real numbers. Then*

$$\int_{-\infty}^{\infty} \frac{1}{\cosh^2 u} \sum_{j=1}^{4} b_j \log |\zeta(1 + \eta + ijt_1 + iut_2)| \, du \ge -2b_0 \log \zeta(1 + \eta).$$

Remark. Lemma 5.1 marks a departure from other treatments, where the bound $|\zeta(1+\eta+iw)| \ge \zeta(1+\eta)^{-1}$ is used at the outset (in the context of a different integral), which in our situation gives

$$I \ge -2(b_1 + \cdots + b_4) \log \zeta(1 + \eta).$$

The new idea is to combine the $\log |\zeta(\cdot)|$ terms using (5.1) to significantly reduce this part of the estimation. The idea in Lemma 6.1 accounts for the majority of the improvement over Heath-Brown's zero-free region. See also the remarks at the end of section 8.

Proof. Denote by I the integral in the lemma. We begin with the Euler product representation for $\zeta(s)$ in the form

$$\log |\zeta(s)| = -\Re \sum_{p} \log(1 - p^{-s}) = \Re \sum_{\substack{p \\ m \ge 1}} \tfrac{1}{m} p^{-ms} \quad (\Re s > 1). \tag{5.3}$$

Next, if $y \ne 0$,

$$U(y) := \int_{-\infty}^{\infty} \frac{e^{iyu}}{\cosh^2 u} \, du = \frac{\pi y}{\sinh(\pi y/2)} \ge 0, \tag{5.4}$$

which can be proved by contour integration. By (5.2), (5.3) and (5.4),

$$\begin{aligned} I &= \sum_{p,m} \tfrac{1}{m} p^{-m(1+\eta)} \Re \left(\sum_{j=1}^{4} b_j p^{-ijmt_1} \int_{-\infty}^{\infty} \frac{p^{-imut_2}}{\cosh^2 u} du \right) \\ &= \sum_{p,m} \tfrac{1}{m} p^{-m(1+\eta)} U(mt_2 \log p) \sum_{j=1}^{4} b_j \cos(jmt \log p) \\ &\ge -b_0 \sum_{p,m} \tfrac{1}{m} p^{-m(1+\eta)} U(mt_2 \log p). \end{aligned}$$

Since $U(y) \le 2$ for all y, we obtain $I \ge -2b_0 \log \zeta(1+\eta)$, as claimed. □

6. The functions f, F and K

Suppose that $t \ge 10000$, $\zeta(\beta + it) = 0$ and λ is a number with $0 < \lambda \le 1 - \beta$ such that

$$\zeta(s) \ne 0 \qquad (1 - \lambda < \Re s \le 1, t - 1 \le \Im s \le 4t + 1). \tag{6.1}$$

Let f be a function with compact support, define F, F_0 and K as in Lemma 4.5, and assume that (4.6) holds. Let a_1, a_2 be real numbers and define $b_0, \dots, b_4$ by (5.2). Put $b_5 = b_1 + b_2 + b_3 + b_4$. By (5.1),

$$\Re \sum_{j=0}^{4} b_j K(1 + ijt) = \sum_{n=1}^{\infty} \Lambda(n) n^{-1} f(\log n) \sum_{j=0}^{4} b_j \cos(jt \log n) \ge 0. \tag{6.2}$$

We next apply Lemma 4.6 with $s = 1$ and $s = 1 + ijt$ $(j = 1, 2, 3, 4)$. Together with Lemma 5.1 (with $t_2 = \frac{2\eta}{\pi}$) and (6.2), this gives

(6.3)

$$\begin{aligned} 0 \le &- \Re \sum_{\substack{1 \le j \le 4 \\ |1+ijt-\rho| \le \eta}} b_j \left(F(1 + ijt - \rho) + f(0) \left(\tfrac{\pi}{2\eta} \cot \left(\tfrac{\pi(1+ijt-\rho)}{2\eta} \right) - \tfrac{1}{1+ijt-\rho} \right) \right) \\ &+ \frac{f(0)}{2\eta} \left[b_5 \left(\tfrac{2}{3} L_2 + B\eta^{3/2} L_1 + \log A \right) + b_0 \log \zeta(1+\eta) \right] + b_0 F(0) \\ &+ D \left(b_5 \left(1.8 + \tfrac{L_1}{3} \right) + 1.8 b_0 + \sum_{j=1}^{4} b_j \sum_{|1+ijt-\rho| \ge \eta} \frac{1}{|1 + ijt - \rho|^2} \right), \end{aligned}$$

where for brevity we write

$$L_1 = \log(4t + 1), \qquad L_2 = \log\log(4t + 1).$$

We choose a function f which is based on the functions given by Lemma 7.5 of [9]. Let θ be the unique solution of

$$\sin^2\theta = \frac{b_1}{b_0}(1-\theta\cot\theta), \quad 0<\theta<\pi/2, \tag{6.4}$$

and define the real function

$$g(u) = \begin{cases} (\cos(u\tan\theta)-\cos\theta)\sec^2\theta & |u| \le \frac{\theta}{\tan\theta}, \\ 0 & \text{else.} \end{cases} \tag{6.5}$$

Set $w(u) = g*g(u)$ (the convolution square of g) for $u \ge 0$ and

$$W(z) = \int_0^\infty e^{-zu} w(u)\,du.$$

From (6.5) we deduce (cf. Lemma 7.1 of [9]) the identities

$$\begin{aligned} W(0) &= 2\sec^2\theta(1-\theta\cot\theta)^2, \\ W(-1) &= 2\tan^2\theta + 3 - 3\theta(\tan\theta+\cot\theta), \\ w(0) &= \sec^2\theta(\theta\tan\theta + 3\theta\cot\theta - 3). \end{aligned} \tag{6.6}$$

Then we take (see (6.1))

$$f(u) = \lambda e^{\lambda u} w(\lambda u) \qquad (u \ge 0) \tag{6.7}$$

and

$$F(z) = \int_0^\infty e^{-zu} f(u)\,du = W\left(\frac{z}{\lambda}-1\right). \tag{6.8}$$

For real y,

$$\Re W(iy) = 2\left(\int_0^\infty w(u)\cos(uy)\,du\right)^2 \ge 0.$$

Since $W(z) \to 0$ uniformly as $|z| \to \infty$ and $\Re z \ge 0$, it follows from the maximum modulus principle (applied to $e^{-W(z)}$) that

$$\Re W(z) \ge 0 \qquad (\Re z \ge 0). \tag{6.9}$$

7. An inequality for the real part of a zero

In this section, we take specific values for a_1 and a_2 and prove the following inequality.

Lemma 7.1. *Suppose $t \ge 10000$, $\zeta(\beta+it)=0$ and* (6.1) *holds. Suppose further that* (1.2) *holds with $B>0$ and $A>6.5$, and that*

$$1-\beta \le \eta/2, \quad 0<\lambda \le \min\left(1-\beta, \tfrac{1}{250}\eta\right). \tag{7.1}$$

Then

$$\frac{1}{\lambda}\left[0.16521 - 0.1876\left(\tfrac{1-\beta}{\lambda} - 1\right)\right] \le 1.471\frac{1-\beta}{\eta^2}$$
$$+ \frac{1}{2\eta}\left[\tfrac{666550}{200211}\left(\tfrac{2}{3}L_2 + B\eta^{3/2}L_1 + \log A\right) + \log\zeta(1+\eta)\right]$$
$$+ 3.683\lambda\left[(6.466 + 5.392B(\eta^{-\frac{1}{2}} - 2))L_1 + 4L_2 + \frac{\frac{\log(A/\eta)+\frac{2}{3}L_2}{1.879} + 0.224}{\eta^2}\right].$$

Proof. A near optimal choice of parameters is $a_1 = 0.225$, $a_2 = 0.9$. By (5.2),

$$b_0 = 10.01055 \qquad b_3 = 4.5,$$
$$b_1 = 17.14500 \qquad b_4 = 1.0,$$
$$b_2 = 10.68250 \qquad b_5 = 33.3275,$$

and by (6.4) and (6.6),

$\theta = 1.15221462997636304887 7\ldots, \quad w(0) = 6.82602968445295450905\ldots.$

The function $W(z)$ has the explicit formula (found with the aid of Maple)

$$W(z) = \frac{w(0)}{z} + W_0(z), \tag{7.2}$$

where

$$W_0(z) = \frac{c_0\left(c_2((z+1)^2e^{-2(\theta/\tan\theta)z} + z^2 - 1) + c_1z + c_3z^3\right)}{z^2(z^2+\tan^2\theta)^2} \tag{7.3}$$

and

$c_0 = \dfrac{1}{\sin\theta\cos^3\theta} = 16.29832162239323505 62\ldots$

$c_1 = (1 + 2(\theta\cos\theta - \sin\theta)\cos\theta)\tan^4\theta = 16.28781036821666318 25\ldots$

$c_2 = \tan^3\theta\sin^2\theta = 9.48131694529505216 82\ldots$

$c_3 = (2 - 5\sin\theta\cos\theta + \theta + 4\theta\cos^2\theta)\tan^2\theta = 10.39249621503336248 95\ldots$

If $R \ge 3$, (7.3) implies

$$|W_0(z)| \le \frac{H(R)}{|z|^3} \qquad (\Re z \ge -1, |z| \ge R), \tag{7.4}$$

where

$$H(R) = \frac{c_0\left(c_2\frac{(R+1)^2}{R^3}\left(e^{2\theta/\tan\theta} + 1\right) + \frac{c_1}{R^2} + c_3\right)}{\left(1 - \frac{\tan^2\theta}{R^2}\right)^2}.$$

By (6.7), (6.8) and (7.2),

$$F_0(z) = F(z) - \frac{f(0)}{z} = W\left(\frac{z}{\lambda} - 1\right) - \frac{\lambda w(0)}{z}$$
$$= W_0\left(\frac{z}{\lambda} - 1\right) + \frac{\lambda f(0)}{z(z-\lambda)}.$$

Suppose $\Re z \ge 0$ and $|z| \ge (R+1)\lambda$. Writing $z' = \frac{z}{\lambda} - 1$, we have $\Re z' \ge -1$ and $|z'| \ge R$. Thus, by (6.7) and (7.4), we obtain

$$|F_0(z)| \le \frac{H(R)\lambda^3}{|z-\lambda|^3} + \frac{w(0)\lambda^2}{|z(z-\lambda)|} \le c_4 \frac{\lambda f(0)}{|z|^2},$$

where

$$c_4 = \frac{H(R)(R+1)^2}{R^3 w(0)} + 1 + 1/R. \tag{7.5}$$

Therefore, providing that $\eta \ge (R+1)\lambda$, (4.6) holds with

$$D = c_4 \lambda f(0). \tag{7.6}$$

Next, define

$$V_c(z) = cw(0)\left(\cot z - \tfrac{1}{z}\right) + W\left(\tfrac{z}{c} - 1\right).$$

By (6.7) and (6.8),

$$F(1+ijt-\rho) + f(0)\left(\tfrac{\pi}{2\eta}\cot\left(\tfrac{\pi(1+ijt-\rho)}{2\eta}\right) - \tfrac{1}{1+ijt-\rho}\right) = V_c(z),$$

where $z = \frac{\pi}{2\eta}(1+ijt-\rho)$ and $c = \frac{\pi\lambda}{2\eta}$. In order to bound the first double sum in (6.3) (leaving only the single term corresponding to $\rho = \beta + it$), we prove that for $0 < c \le \frac{\pi}{2R+2}$,

$$\Re V_c(z) \ge -c_5 c^2 w(0) \qquad \left(\Re z \ge c, |z| \le \tfrac{\pi}{2}\right).,\tag{7.7}$$

where

$$c_5 = \frac{4}{\pi^2}\left(1 + \frac{(R+1)^2 H(R)}{w(0)R^3}\right) = \frac{4}{\pi^2}(c_4 - 1/R).\tag{7.8}$$

By the maximum modulus principle (applied to $e^{-V_c(z)}$), it suffices to prove (7.7) on the boundary of the region. First consider z satisfying $\Re z = c$, $|z| \le \pi/2$. By Lemma 4.4 and (6.9),

$$\Re V_c(z) \ge cw(0)\Re\left(\cot z - \frac{1}{z}\right) \ge -\frac{4c^2 w(0)}{\pi^2}.$$

When $|z| = \pi/2$ and $x = \Re z \geq c$, we have $|z/c - 1| \geq R$, so by (7.4), $|W_0(z/c-1)| \leq H(R)|z/c-1|^{-3}$. Thus, by (7.2) and Lemma 4.4,

$$\begin{aligned}\Re V_c(z) &\geq -\frac{4cw(0)x}{\pi^2} + \frac{cw(0)(x-c)}{|z-c|^2} - \frac{H(R)c^3}{|z-c|^3} \\ &\geq -\frac{4cw(0)x}{\pi^2} + \frac{cw(0)(x-c)}{(\pi/2)^2} - \frac{H(R)c^3}{(\pi/2-c)^3} \\ &= c^2 w(0)\left(-\frac{4}{\pi^2} - \frac{H(R)c}{w(0)(\pi/2-c)^3}\right).\end{aligned}$$

Noting that $c \leq \frac{\pi}{2R+2}$ completes the proof of (7.7). In fact, with more work one can prove that (7.7) holds with $c_5 = \frac{1}{3}$.

By (7.7), we have

$$\begin{aligned}&-\Re \sum_{\substack{1\leq j\leq 4 \\ |1+ijt-\rho|\leq\eta}} b_j \left(F(1+ijt-\rho) + f(0)\left(\tfrac{\pi}{2\eta}\cot\left(\tfrac{\pi(1+ijt-\rho)}{2\eta}\right) - \tfrac{1}{1+ijt-\rho}\right)\right) \\ &\quad\leq -b_1 V_c(\tfrac{\pi}{2\eta}(1-\beta)) + c_5 c^2 w(0) \sum_{j=1}^{4} b_j N(jt,\eta).\end{aligned}$$

Combining this last estimate with (6.3), (6.7), (7.6) and Lemma 4.3 gives

$$\begin{aligned}0 &\leq b_0 F(0) - b_1 V_c\left(\tfrac{\pi}{2\eta}(1-\beta)\right) + \frac{\lambda f(0)}{\eta^2}\left(\tfrac{\pi^2 c_5}{4} - c_4\right)\sum_{j=1}^{4} b_j N(jt,\eta) \\ &\quad + \frac{f(0)}{2\eta}\left[b_5\left(\tfrac{2}{3}L_2 + B\eta^{3/2}L_1 + \log A\right) + b_0 \log\zeta(1+\eta)\right] \\ &\quad + c_4\lambda f(0) b_5 \Bigg[1.8 + \tfrac{L_1}{3} + 1.8\tfrac{b_0}{b_5} + \left(6.132 + 5.392B(\eta^{-\frac{1}{2}} - 2)\right)L_1 \\ &\quad + 13.5 - 8.5\log A + 4L_2 + \frac{1}{\eta^2}\left(\frac{\log A - \log\eta + \frac{2}{3}L_2}{1.879} + 0.224\right)\Bigg].\end{aligned} \tag{7.9}$$

The sum in (7.9) can be ignored because of (7.8). Also, by the lower bound on A we have

$$1.8 + 1.8\tfrac{b_0}{b_5} + 13.5 - 8.5\log A < 0. \tag{7.10}$$

Put $R = 249$, and compute $H(249) \leq 171.8$ and $c_4 \leq 1.106$. Since $\cot x - \frac{1}{x} \geq -0.348x$ for $0 < x \leq \frac{\pi}{4}$ and $1-\beta \leq \frac{1}{2}\eta$, we have

$$V_c\left(\tfrac{\pi}{2\eta}(1-\beta)\right) \geq F(1-\beta) - 0.348 f(0)\tfrac{\pi^2}{4\eta^2}(1-\beta). \tag{7.11}$$

By (6.6), (6.7) and (6.8),

(7.12)

$$-\frac{b_1}{b_0}F(1-\beta)+F(0) = -\left(\frac{b_1}{b_0}W\left(\tfrac{1-\beta}{\lambda}-1\right)-W(-1)\right)$$
$$= -\left(\frac{b_1}{b_0}W(0)-W(-1)\right)+\frac{b_1}{b_0}\left(W(0)-W\left(\tfrac{1-\beta}{\lambda}-1\right)\right)$$
$$= \frac{-f(0)\cos^2\theta}{\lambda}+\frac{b_1}{b_0}\left(W(0)-W\left(\tfrac{1-\beta}{\lambda}-1\right)\right).$$

Since $W(x)$ and $W'(x)$ are both decreasing, we have

$$W(0)-W\left(\tfrac{1-\beta}{\lambda}-1\right) \le \left(\tfrac{1-\beta}{\lambda}-1\right)W'(0) \le 0.7475\left(\tfrac{1-\beta}{\lambda}-1\right).$$

Thus, by (7.11) and (7.12),

(7.13)

$$F(0)-\frac{b_1}{b_0}V_c\left(\tfrac{\pi}{2\eta}(1-\beta)\right) \le 0.348f(0)\frac{\pi^2}{4\eta^2}\frac{b_1}{b_0}(1-\beta)$$
$$+ \quad \frac{f(0)}{\lambda}\left(-\cos^2\theta+\frac{0.7475b_1}{b_0w(0)}\left(\tfrac{1-\beta}{\lambda}-1\right)\right).$$

Dividing both sides of (7.9) by $b_0f(0)$ and using (7.10), (7.13) and the numerical values of b_0, b_1, b_5 and θ completes the proof of the lemma. □

8. The proof of Theorem 2

Suppose T_0 satisfies the hypotheses of Theorem 2 and let

(8.1) $$M = \inf_{\substack{\zeta(\beta+it)=0\\ t\ge T_0}} Z(\beta,t), \quad Z(\beta,t) := (1-\beta)(B\log t)^{\frac{2}{3}}(\log\log t)^{\frac{1}{3}}.$$

By the Korobov-Vinogradov theorem, $M>0$. If $M \ge M_1$, then the theorem is immediate. Otherwise, suppose that $M < M_1 \le 0.05507$. Then there is a zero $\beta+it$ of $\zeta(s)$ with $t\ge T_0$ and

$$Z(\beta,t)\in[M, M(1+\delta)], \quad \delta = \min\left(\tfrac{10^{-100}}{\log T_0}, \tfrac{M_1-M}{2M}\right).$$

By (8.1), (6.1) holds with

(8.2) $$\lambda = ML_1^{-2/3}L_2^{-1/3}B^{-2/3}.$$

Again we make the abbreviations $L_1=\log(4t+1)$, $L_2=\log\log(4t+1)$. Define b_0, b_5 as in the previous section. We apply Lemma 7.1, taking

(8.3) $$\eta = EB^{-\frac{2}{3}}\left(\frac{L_2}{L_1}\right)^{\frac{2}{3}}, \quad E = \left(\frac{4(1+b_0/b_5)}{3}\right)^{\frac{2}{3}} = \left(\frac{1733522}{999825}\right)^{\frac{2}{3}}.$$

The lower bound $\frac{\log T_0}{\log\log T_0} \ge \frac{1740}{B}$ ensures that $\eta \le 0.01$ and

$$\lambda \le 0.5507(BL_1)^{-\frac{2}{3}} L_2^{-\frac{1}{3}} \le \frac{\eta}{250}.$$

The inequalities $T_0 \ge e^{30000}$ and $M_1 \le 0.05507$ ensure that the other hypotheses of Lemma 7.1 are met. In addition,

$$\frac{1-\beta}{\lambda} - 1 \le (1+\delta)\left(\frac{L_1}{\log t}\right)^{\frac{2}{3}} \left(\frac{L_2}{\log\log t}\right)^{\frac{1}{3}} - 1 \le \frac{0.97}{\log T_0}. \tag{8.4}$$

By Lemma 3.1,

$$\log\zeta(1+\eta) \le \log(1/\eta + 0.6) \le \log(1/\eta) + 0.006. \tag{8.5}$$

We now apply Lemma 7.1, using the upper bounds for $(1-\beta)$ and λ on the right side of the conclusion. First, since $-\log\eta \approx \frac{2}{3}L_2$, we have by (8.3),

$$\begin{aligned} \frac{1}{2\eta}\left[\frac{b_5}{b_0}\left(\frac{2L_2}{3} + B\eta^{\frac{3}{2}}L_1\right) + \frac{2L_2}{3}\right] &= \frac{b_5}{b_0}\left(1+\frac{b_0}{b_5}\right)^{\frac{1}{3}}\left(\frac{3B}{4}\right)^{\frac{2}{3}} L_1^{\frac{2}{3}} L_2^{\frac{1}{3}} \\ &\le 2.99968(BL_1)^{\frac{2}{3}} L_2^{\frac{1}{3}}. \end{aligned} \tag{8.6}$$

This constitutes the main term as $t \to \infty$. Next, since $Z(\beta, t) \le M_1$ and by the lower bound on T_0,

$$1.471\frac{1-\beta}{\eta^2} \le 0.039 B^{2/3} L_1^{2/3} L_2^{-5/3} \le 0.0038 B^{2/3}\left(\frac{L_1}{L_2}\right)^{2/3}. \tag{8.7}$$

Using (8.5), the remaining part of the second line in the conclusion of Lemma 7.1 is

$$\begin{aligned} &\le \frac{1}{2\eta}\left[\tfrac{b_5}{b_0}\log A - \log E + \tfrac{2}{3}\log(B/L_2) + 0.006\right] \\ &\le \frac{B^{\frac{2}{3}}}{2E}\left(\frac{L_1}{L_2}\right)^{\frac{2}{3}}\left[3.3293\log A - 0.3608 + \tfrac{2}{3}\log(B/L_2)\right] \\ &\le \left(\frac{BL_1}{L_2}\right)^{\frac{2}{3}}\left(1.1534\log A - 0.125 + 0.2310\log(B/L_2)\right). \end{aligned} \tag{8.8}$$

By (8.2), (8.3), and $L_2 \le 0.00035 L_1$, the third line in the conclusion of Lemma 7.1 is

(8.9)

$$
\begin{aligned}
&\le 0.2029 L_1^{-\frac{2}{3}} L_2^{-\frac{1}{3}} B^{-\frac{2}{3}} \Bigg[\Bigg(6.468 + \frac{5.392 B^{\frac{4}{3}}}{\sqrt{E}} \left(\frac{L_1}{L_2} \right)^{\frac{1}{3}} - 10.784 B \Bigg) L_1 \\
&\qquad + \frac{B^{\frac{4}{3}}}{E^2} \left(\frac{L_1}{L_2} \right)^{\frac{4}{3}} \left(\frac{\log A + \frac{4}{3} L_2 + \frac{2}{3} \log(B/L_2) - \log E}{1.879} + 0.224 \right) \Bigg] \\
&\le \left(\frac{B L_1}{L_2} \right)^{\frac{2}{3}} \Bigg[0.9798 + \frac{1.313 - 2.188 B}{B^{\frac{4}{3}}} \left(\frac{L_2}{L_1} \right)^{\frac{1}{3}} \\
&\qquad + \frac{0.05185}{L_2} \left(\log A + \tfrac{2}{3} \log(B/L_2) + 0.05401 \right) \Bigg] \\
&\le \left(\frac{B L_1}{L_2} \right)^{\frac{2}{3}} \left[\frac{1.313 - 2.188 B}{B^{4/3}} \left(\frac{L_2}{L_1} \right)^{\frac{1}{3}} + 0.0051 (\log A + \tfrac{2}{3} \log(\tfrac{B}{L_2})) + 0.9801 \right]
\end{aligned}
$$

Combining (8.4)–(8.9) with Lemma 7.1 gives

$$
\frac{1}{\lambda} \left(0.16521 - \frac{0.182}{\log T_0} \right) \le (B L_1)^{\frac{2}{3}} L_2^{\frac{1}{3}} \left(2.99968 + \tfrac{X(t)}{L_2} \right).
$$

By (8.2), this gives

$$
M \ge \frac{0.16521 - 0.182/\log T_0}{2.99968 + X(t)/\log\log t} \ge M_1.
$$

This concludes the proof of Theorem 2.

Remarks. Compared with the methods in [8], there are two improvements evident in (8.6). First, the factor $(3/4)^{2/3} \approx 0.82548$ replaces the factor $2^{-1/3} K_2 \approx 0.843445$ from ([8], p. 197). This improvement comes from integrating over two vertical lines (Lemma 2.2). The second and larger improvement is the factor $(1 + b_0/b_5)^{1/3}$, which is $2^{1/3}$ in the treatment of [8], and comes from combining the $\log |\zeta(\cdot)|$ terms in Lemma 5.1. Together these improve the bounds from [8] by about 17%.

9. The proof of Theorem 3

Almost everything in Sections 2–6 is identical. In place of (1.2) we use an explicit form of the Van der Corput bound (1.6). We fix $\eta = \frac{1}{2}$, and the proof of Lemma 5.1 gives

(9.1)
$$-\int_{-\infty}^{\infty} \frac{\sum_{j=1}^4 b_j \log|\zeta\left(\frac{3}{2}+ijt+\frac{iu}{\pi}\right)|}{\cosh^2 u}\,du \le b_0 \sum_{p,m} \tfrac{1}{m} p^{-\frac{3}{2}m} U(\tfrac{m}{\pi}\log p)$$
$$= 2b_0 \sum_{p,m} \frac{\log p}{p^{2m}-p^m}$$
$$= 2b_0 \sum_{n=2}^{\infty} \frac{\Lambda(n)}{n^2-n}$$
$$\le 1.702 b_0.$$

Let $T_0 = 545000000$ and suppose that $\zeta(\beta+it)=0$ with $t \ge T_0$ (it is known that all zeros with $|t| < T_0$ have real part $\frac{1}{2}$ [14]). In place of Lemma 3.4 we use

Lemma 9.1. *If* $t \ge T_0$, *then*

$$I(t) = \int_{-\infty}^{\infty} \frac{\log|\zeta(1/2+it+iu/\pi)|}{\cosh^2 u}\,du \le 2J(t),$$

where $J(t)$ *is given by* (1.7).

Proof. From (1.6), $|\zeta(1/2+it)| \le 3t^{1/6}\log t$ for $t \ge 3$, so by Lemma 3.4, $I(t) \le 2(\frac{1}{6}\log t + \log\log t + \log 3)$. Using the first inequality from (1.6), we have

$$I(y) \le \int_{-\infty}^{\infty} \frac{\log(57+6(t+|u|)^{1/4})}{\cosh^2 u}\,du = 2\int_0^{\infty} \frac{\log(57+6(t+u)^{1/4})}{\cosh^2 u}\,du.$$

When $0 \le u \le \log t$, the numerator is $\le \log(6.37306t^{1/4})$ and when $u > \log t$, the numerator is $\le \log(6.4(e^u+u)^{1/4}) \le u$ and the denominator is $\ge \frac{1}{4}e^{2u}$. Therefore,

$$I(t) \le 2\log(6.37306t^{1/4}) + 8\int_{\log t}^{\infty} ue^{-2u}\,du \le \frac{\log t}{2} + 3.7042.$$

□

We make the assumption (6.1) as before and take the same values for a_1, a_2 (so $b_0, \ldots, b_4$, θ, w, f, F, W are the same as in section 7). The only change in (6.3) is that the term $\frac{2}{3}\log\log t + B\eta^{3/2}\log t + \log A$ is replaced by $J(t)$. Next, we follow the proof of Lemma 7.1. Using (4.3) (Rosser's theorem) as in the proof of Lemma 4.3, we obtain for $t \ge 10000$

$$\sum_{|1+ijt-\rho|\ge \frac12} \frac{1}{|1+ijt-\rho|^2} \le 3.2357\log t + 5.316 \log\log t + 16.134 - 4N(t,1/2). \tag{9.2}$$

Assume that

$$0 < \lambda \le 1-\beta \le \frac{1}{160}. \tag{9.3}$$

Let $R = \frac{1}{2(1-\beta)} - 1 \ge 79$. By (9.3), $\eta \le 80\lambda$. As in the proof of (7.5), we deduce that (4.6) holds with

$$D = c_4 \lambda f(0), \quad c_4 = \frac{H(79)(R+1)^2}{R^3 w(0)} + 1 + \frac{1}{R} \le 1.35. \tag{9.4}$$

Also, (7.7) is replaced by

$$\Re V_c(z) \ge -c_5 c^2 w(0) = -c_5 \pi^2 \lambda f(0) \quad (\Re z \ge c, |z| \le \pi/2), \tag{9.5}$$

valid for $0 < c \le \pi(1-\beta)$ with

$$c_5 = \frac{4}{\pi^2} + \frac{\pi(1-\beta)H(79)}{w(0)(\pi/2 - \pi(1-\beta))^2}. \tag{9.6}$$

Analogously to (7.9), the inequalities (9.1), (9.2), (9.4), and (9.5) give

(9.7)

$$\begin{aligned} 0 \le\ & b_0 F(0) - b_1 V_{\pi\lambda}(\pi(1-\beta)) + (\pi^2 c_5 - 4c_4)\lambda f(0) \sum_{j=1}^4 b_j N(jt, \tfrac12) \\ & + f(0)\left(b_5 J(4t+1) + 0.851 b_0\right) \\ & + 1.35\lambda f(0)\left[b_5(1.8 + \tfrac{L_1}{3}) + 1.8 b_0 + b_5(3.2357 L_1 + 5.316 L_2 + 16.134)\right]. \end{aligned}$$

As before we use $L_1 = \log(4t+1)$, $L_2 = \log\log(4t+1)$. By (9.4) and (9.6), $\pi^2 c_5 - 4c_4 = -4/R < 0$, so the sum in (9.7) can be ignored. By (9.3), $\cot x - 1/x \ge -0.3334x$ for $0 < x \le \pi(1-\beta)$ and this gives

$$V_{\pi\lambda}(\pi(1-\beta)) \ge F(1-\beta) - 0.3334\pi^2(1-\beta) f(0).$$

By an argument similar to that leading to (7.13), we obtain

$$\begin{aligned} F(0) - \frac{b_1}{b_0} V_{\pi\lambda}(\pi(1-\beta)) \le\ & 0.3334\pi^2(1-\beta)\frac{b_1}{b_0} f(0) \\ & + \frac{f(0)}{\lambda}\left(-\cos^2\theta + \frac{0.7475 b_1}{b_0 w(0)}\left(\frac{1-\beta}{\lambda} - 1\right)\right) \end{aligned} \tag{9.8}$$

Combining (9.8) with (9.7) gives the following bound.

Lemma 9.2. *Suppose that $\zeta(\beta+it)=0$ with $t \ge 545000000$ and $1-\beta \le \frac{1}{160}$. Let λ be a positive number satisfying* (6.1). *Then*

$$\frac{0.16521-0.1876(\frac{1-\beta}{\lambda}-1)}{\lambda} \le 5.646(1-\beta)+\frac{b_5}{b_0}J(4t+1) + 0.851 + 1.35\lambda\frac{b_5}{b_0}(3.5691L_1+5.316L_2+18.475). \tag{9.9}$$

To prove Theorem 3, first define

$$c_6 = c_6(t) = \frac{1}{J(t)+1.15}. \tag{9.10}$$

For a zero $\beta+it$ of ζ with $t \ge T_0$, define $Y(\beta,t)$ by the equation

$$1-\beta = \frac{0.04962-0.0196c_6(t)}{J(t)+Y(\beta,t)}.$$

By the Korobov-Vinogradov theorem, $Y(\beta,t) \to -\infty$ as $t \to \infty$. Let $M = \max_{t\ge T_0} Y(\beta,t)$. If $M \le 1.15$, Theorem 3 follows. Otherwise, suppose $\beta+it$ is a zero with $Y(\beta,t)=M>1.15$. Then (6.1) holds with

$$\lambda = \frac{0.04962-0.0196c_6(t)}{J(4t+1)+M} \tag{9.11}$$

By (9.10) and (9.11),

$$\frac{1-\beta}{\lambda}-1 = \frac{J(4t+1)+M}{J(t)+M}-1 = \frac{J(4t+1)-J(t)}{J(t)+M} \le \frac{0.3466}{J(t)+M}. \tag{9.12}$$

Apply Lemma 9.2, multiplying both sides of (9.9) by $6b_0/b_5$. By (9.12), the left side is

$$\ge \frac{0.04962-0.0196c_6(t)}{\lambda} = J(4t+1)+M.$$

Using $L_1 \le \log t + \log 4 + \frac{1}{4t}$ and $L_2 \le \log\log t + (\log 4 + \frac{1}{4t})/\log t$, we conclude that

$$\begin{aligned} M &\le 0.25562 + (1-\beta)\left[1.696+1.35(3.5691L_1+5.316L_2+18.475)\right] \\ &\le 0.25562+(1-\beta)\left[33.812+4.8183\log t+7.1766\log\log t\right]. \end{aligned} \tag{9.13}$$

Also, by assumption $1-\beta \le (0.04962-0.0196c_6(t))c_6(t)$. Plugging this into (9.13), and using a short Maple computation, we find that

$$M \le 0.685+0.155\log\log t.$$

In fact, $M \le 1.7$ for all $t \ge T_0$, but the above bound suffices for our purposes. This completes the proof of Theorem 3.

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