VINOGRADOV'S INTEGRAL AND BOUNDS FOR THE RIEMANN ZETA FUNCTION

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1. Introduction

The methods of Korobov [11] and Vinogradov [28] produce a zero-free region for the Riemann zeta function $\zeta(s)$ of the following strength: for some $c > 0$, there are no zeros of $\zeta(s)$ for $s = \sigma + it$ with $|t| \ge 3$ and $\sigma > 1 - c(\log|t|)^{-2/3}(\log\log|t|)^{-1/3}$. The principal tool is an upper bound for $|\zeta(s)|$ near the line $\sigma = 1$. In 1967, Richert [22] used this method to give the bound

$$|\zeta(\sigma+it)| \le A|t|^{B(1-\sigma)^{3/2}} \log^{2/3}|t| \qquad (|t| \ge 2, \tfrac12 \le \sigma \le 1) \tag{1.1}$$

with $B = 100$ and A and unspecified absolute constant. Similar results with smaller B values have been proven subsequently by several authors, the best being $B = 18.4974$ and due to Kulas [13]. Recently, Y. Cheng [3] has given a completely explicit version of this bound, with $A = 175$ and $B = 46$.

In this paper, we improve substantially the value of B, while also keeping the bound entirely explicit. More generally, we bound the Hurwitz zeta function, defined for $\Re s > 1$ and $0 < u \le 1$ by $\zeta(s,u) = \sum_{n=0}^\infty (n+u)^{-s}$. The Hurwitz zeta function may be used to bound Dirichlet L-functions via the identity $L(s,\chi) = q^{-s}\sum_{m=1}^q \chi(m)\zeta(s,m/q)$, where χ is a Dirichlet character modulo q. Notice that $\zeta(s) = \zeta(s,1)$. Since $\zeta(\bar s, u) = \overline{\zeta(s,u)}$, we may restrict our attention to s lying in the upper half-plane.

Theorem 1. *The inequalities*

$$|\zeta(\sigma+it)| \le At^{B(1-\sigma)^{3/2}}\log^{2/3} t \qquad (t \ge 3, \tfrac12 \le \sigma \le 1),$$
$$|\zeta(\sigma+it,u) - u^{-s}| \le At^{B(1-\sigma)^{3/2}}\log^{2/3} t \qquad (0 < u \le 1, t \ge 3, \tfrac12 \le \sigma \le 1)$$

hold with $B = 4.45$ and $A = 76.2$.

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If the Riemann Hypothesis is true, then the conclusion of Theorem 1 holds with any positive B, with the constant A depending on B. Bounds of the type (1.1) with explicit values of B have numerous applications, including (i) explicit zero-free regions for $\zeta(s)$; (ii) explicit error bounds for the prime number theorem; (iii) zero density bounds for $\zeta(s)$; (iv) mean value theorems for $\zeta(s)$; (v) bounds for error terms in the Dirichlet divisor problem. We briefly indicate the consequences of Theorem 1 for each of these five problems.

(i) One can use (1.1) to give explicit values for the constant c in the zero-free region mentioned in the opening paragraph. In a separate paper [6], the author shows that $\zeta(\beta+it)\neq 0$ for t sufficiently large and

$$1-\beta \le \frac{0.05507 B^{-2/3}}{(\log t)^{2/3}(\log\log t)^{1/3}}.$$

Moreover, using the full strength of Theorem 1, in [6] the zero-free region

$$t\ge 3, \quad 1-\beta \le \frac{c}{(\log t)^{2/3}(\log\log t)^{1/3}}, \quad c=\frac{1}{57.54}$$

is proved. By comparison, Popov [20] showed that the above holds with holds with $c = 0.00006888$, and Cheng [4] proved a zero-free region with $c = 1/990$.

(ii) A corollary of Theorem 1, the work in [6], and Theorem 8 of Pintz [19], is the following error bound in the prime number theorem:

$$\pi(x)-\mathrm{li}(x) = O\left(x\exp\{-c(\log x)^{3/5}(\log\log x)^{-1/5}\}\right), \quad c=0.2098.$$

(iii) Let $N(\sigma,T)$ denote the number of zeros of $\zeta(s)$ in the rectangle $\sigma\le\Re s\le 1$, $|\Im s|\le T$. If (1.1) holds, then for $\frac{9}{10}\le\sigma\le 1$, we have

$$N(\sigma,T)\ll T^{13.043B(1-\sigma)^{3/2}}\log^{15} T.$$

This follows from Theorem 12.3 of Montgomery [17], taking $1-\alpha = 4.93(1-\sigma)$; see also §11.4 of [8]. Incidentally, there is an error in Corollary 12.5 of [17], where it is stated that $B = 100$ implies

$$N(\sigma,T)\ll T^{167(1-\sigma)^{3/2}}\log^{17} T.$$

As a corollary, Theorem 1 gives

$$N(\sigma,T)\ll T^{58.05(1-\sigma)^{3/2}}\log^{15} T.$$

(iv) Let

$$M_k(\sigma,T) = \frac{1}{T}\int_0^T |\zeta(\sigma+it)|^{2k}\,dt.$$

Let σ_k be the infimum of the numbers σ with $M_k(\sigma, T) = O(1)$, and let $\mu_k(\sigma)$ be the infimum of the numbers ξ such that $M_k(\sigma, T) = O(T^{\xi})$. If $\sigma > \sigma_k$, we have an asymptotic formula for $M_k(\sigma, T)$ ([25], §7.8):

$$M_k(\sigma, T) \sim \sum_{n=1}^{\infty} \frac{d_k(n)^2}{n^{2\sigma}},$$

where $d_k(n)$ is the number of k-tuples of positive integers $(b_1, b_2, \cdots, b_k)$ with $b_1 \cdots b_k = n$. In particular, $d_2(n)$ is the number of positive divisors of n. Also, when $\Re s > 1$, $(\zeta(s))^k = \sum_{n=1}^{\infty} d_k(n) n^{-s}$. Upper bounds on σ_k can be deduced from upper bounds on $\zeta(s)$ inside the critical strip by means of a Theorem of Carlson ([25], Theorem 7.9): for any $0 < \alpha < 1$, we have

$$\sigma_k \le \max\left(\frac{1}{2}, \alpha, 1 - \frac{1-\alpha}{1+\mu_k(\alpha)}\right). \tag{1.2}$$

By (1.1), we have trivially $\mu_k(\sigma) \le 2Bk(1-\sigma)^{3/2}$. Taking $\alpha = 1 - (Bk)^{-2/3}$ in (1.2) gives $\sigma_k \le 1 - \frac{1}{3}(Bk)^{-2/3}$. For more on mean value theorems, see Chapter VII of [25] and Chapter 8 of [8].

(v) Denote by $\Delta_k(x)$ the usual error term in the Dirichlet divisor problem, i.e.

$$\Delta_k(x) = \sum_{n \le x} d_k(n) - \operatorname*{Res}_{s=1} x^s (\zeta(s))^k s^{-1} = \sum_{n \le x} d_k(n) - x P_k(\log x),$$

where P_k is a certain polynomial. Let α_k be the infimum of numbers α with $\Delta_k(x) = O(x^{\alpha})$. Dirichlet in 1849 proved that $\alpha_2 \le \frac{1}{2}$ and his method can be used to deduce $\alpha_k \le 1 - \frac{1}{k}$. Modern treatments make use of Perron's formula in the form

$$\sum_{n \le x} d_k(n) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \zeta^k(s) \frac{x^s}{s}\, ds, \quad c > 1.$$

Then the contour is moved inside the critical strip, the main term coming from the pole at $s = 1$, and the error term coming from upper bounds for $\zeta(s)$. In 1960, Richert [21] proved that $\alpha_k \le 1 - ck^{-2/3}$ for some positive constant c. Subsequently, the value of c was made explicit as a function of the constant B in (1.1) by Karatsuba [10] ($c = \frac{1}{2}(2B)^{-2/3} \approx 0.31498 B^{-2/3}$). Writing $c = dB^{-2/3}$, the value of d was improved by Ivić and Ouellet [9] to $d = \frac{1}{3} 2^{2/3} \approx 0.52913$. There are two claims for larger d, but both arguments are flawed. Fujii [7] claims $d = 2^{-1/2}(\sqrt{8}-1)^{-1/3} \approx 0.57826$, but the details are omitted (the method appears to give $d = \frac{1}{2}$); Panteleeva [18] claims $d = 2^{-2/3} \approx 0.62996$, but the proof of this result (Theorem 3 of [18]) has a flaw, namely the differentiation of (14) in invalid.

For the mean square of $\Delta_k(x)$, Ivić and Ouellet [9] proved that

$$\int_1^x \Delta_k^2(y)\, dy \ll_{\varepsilon,k} x^{1+2b_k+\varepsilon}, \qquad b_k = 1 - \frac{2}{3}\left(\frac{1}{Bk}\right)^{2/3}.$$

More information may be found in Chapter XII of [25] and Chapter 13 of [8].

Theorem 1 depends primarily on upper bounds for the following exponential sum:

$$S(N,t)=\max_{0<u\le 1}\max_{N<R\le 2N}\left|\sum_{N<n\le R}(n+u)^{-it}\right|,$$

where N is a positive integer and $t\ge N$. We shall prove the following.

Theorem 2. *Suppose N is a positive integer, $N\le t$ and set $\lambda=\frac{\log t}{\log N}$. Then*

$$S(N,t)\le 9.463N^{1-1/(133.66\lambda^2)}.$$

By comparison, Kulas [12] proved that $S(N,t)\ll N^{1-1/(2309.525\lambda^2)}$ for $\lambda\ge 1000$.

Corollary 2A. *Suppose χ is a Dirichlet character modulo q, where $q\le N$ and $2\le N\le qt$. Then*

$$\max_{N<R\le 2N}\left|\sum_{N<n\le R}\chi(n)n^{-it}\right|\le 10.463\frac{\phi(q)}{q}Ne^{-\frac{\log^3(N/q)}{133.66\log^2 t}}.$$

Proof. Suppose the maximum on the left occurs at $R=R_0$. Then

$$\sum_{N<n\le R_0}\chi(n)n^{-it}=\sum_{\substack{\ell=1\\(\ell,q)=1}}^{q}\chi(\ell)\sum_{\substack{N<n\le R_0\\ n\equiv\ell \pmod q}}n^{-it}.$$

Writing $n=mq+\ell$ gives

$$\begin{aligned}\left|\sum_{\substack{N<n\le R_0\\ n\equiv\ell \pmod q}}n^{-it}\right|&\le 1+\left|\sum_{\frac{N-\ell+q}{q}<m\le\frac{R_0-\ell}{q}}(m+\ell/q)^{-it}\right|\\&\le 1+S\left(\frac{N-\ell+q}{q},t\right).\end{aligned}$$

Theorem 2 then gives

$$\left|\sum_{N<n\le R_0}\chi(n)n^{-it}\right|\le\phi(q)\left(1+9.463\left(\frac{N}{q}\right)^{1-\frac{\log^2(N/q)}{133.66\log^2 t}}\right).$$

Lastly, $N/q\ge 1$, and the result follows. □

As with prior treatments, Theorem 2 in turn depends on explicit bounds for Vinogradov's integral, defined as

$$J_{s,k}(P) = \int_{[0,1]^k} \left| \sum_{1\le x\le P} e(\alpha_1 x + \cdots + \alpha_k x^k) \right|^{2s} d\boldsymbol{\alpha}, \tag{1.3}$$

where $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_k)$ and $e(z) = e^{2\pi i z}$. Equivalently, $J_{s,k}(P)$ is the number of solutions of the simultaneous equations

$$\sum_{i=1}^{s} (x_i^j - y_i^j) = 0 \qquad (1 \le j \le k); \quad 1 \le x_i, y_i \le P. \tag{1.4}$$

For $\boldsymbol{h} = (h_1, \dots, h_k)$, let $J_{s,k}(P; \boldsymbol{h})$ be the number of solutions of

$$\sum_{i=1}^{s} (x_i^j - y_i^j) = h_j \quad (1 \le j \le k); \quad 1 \le x_i, y_i \le P.$$

In particular,

$$\begin{aligned} J_{s,k}(P; \boldsymbol{h}) &= \int_{[0,1]^k} \left| \sum_{1\le x\le P} e(\alpha_1 x + \cdots + \alpha_k x^k) \right|^{2s} e(-\alpha_1 h_1 - \cdots - \alpha_k h_k) d\boldsymbol{\alpha} \\ &\le J_{s,k}(P; (0, \dots, 0)) = J_{s,k}(P). \end{aligned}$$

Hence, writing $Q = \lfloor P \rfloor$, we obtain

$$Q^{2s} = \sum_{\boldsymbol{h}} J_{s,k}(P; \boldsymbol{h}) \le \sum_{\substack{\boldsymbol{h} \\ |h_j| \le s(Q^j - 1)}} J_{s,k}(P) \le (2s)^k Q^{k(k+1)/2} J_{s,k}(P).$$

Also, counting only the solutions of (1.4) with $x_i = y_i$ for each i gives $J_{s,k}(P) \ge Q^s$. Therefore

$$J_{s,k}(P) \ge \max\left((2s)^{-k} \lfloor P \rfloor^{2s - \frac{1}{2}k(k+1)}, \lfloor P \rfloor^s \right). \tag{1.5}$$

Upper bounds take the form of

$$J_{s,k}(P) \le D(s,k) P^{2s - \frac{1}{2}k(k+1) + \eta(s,k)}, \tag{1.6}$$

where $\eta(s,k) \ge 0$ and $D(s,k)$ is independent of P. Stechkin in 1975 [24] proved (1.6) with

$$\eta(rk, k) = \frac{1}{2} k^2 (1 - 1/k)^r, \qquad D(rk, k) = \exp\{C \min(r, k) k^2 \log k\}$$

for an absolute constant C. The constant factor was improved by Wooley [31]. Small improvements to the exponents of P were subsequently made by Arkhipov and Karatsuba [1] and Tyrina [26] (significant for $s \ll k^2$). Also significant is Wooley's [32] result when $s \ll k^{3/2-\varepsilon}$, which is very close to the "ideal" bounds $C(k,s)P^s$ in that range of s. For our purposes, the most important improvement comes from Wooley [30], who improved the exponents substantially in a wide range of s, showing that (1.6) holds with $\eta(k,s) \approx \frac12 k^2 e^{1/2-2s/k^2}$ valid for $s \ll k^2 \log k$ (see [5], Lemma 5.2). In Theorem 3 below, we combine Wooley's method with the main idea from [1] to improve this to $\eta(k,s) \approx \frac38 k^2 e^{1/2-2s/k^2}$. In the application to bounding the Riemann zeta function, we will take s to be of order k^2, so this small improvement is significant.

Theorem 3. *Let k and s be integers with $k \ge 1000$ and $2k^2 \le s \le \frac{k^2}{2}(\frac12 + \log \frac{3k}{8})$. Then*

$$J_{s,k}(P) \le k^{2.055k^3-5.91k^2+3s} 1.06^{sk+2s^2/k-9.7278k^3} P^{2s-\frac12 k(k+1)+\Delta_s} \qquad (P \ge 1),$$

where

$$\Delta_s = \tfrac38 k^2 e^{1/2-2s/k^2+1.7/k}.$$

Further, if $k \ge 129$, there is an integer $s \le \rho k^2$ such that for $P \ge 1$,

$$J_{s,k}(P) \le k^{\theta k^3} P^{2s-\frac12 k(k+1)+0.001k^2},$$

with

$$(\rho,\theta) = \begin{cases} (3.21432, 2.3291) & (k \ge 200) \\ (3.21734, 2.3849) & (150 \le k \le 199) \\ (3.22313, 2.4183) & (129 \le k \le 149) \end{cases} \tag{1.7}$$

By itself, Theorem 3 implies the inequalities in Theorem 1 with B a bit more than 10.4.

The most significant new idea is to bound $S(N,t)$ in terms of both $J_{s,k}(P)$ and another quantity which counts the number of solutions of *incomplete* Diophantine systems (where we regard (1.4) to be *complete* because the powers of the variables range from 1 to k). Define $J_{s,k,h}(\mathscr{B})$ to be the number of solutions of the system

$$\sum_{i=1}^{s} (x_i^j - y_i^j) = 0 \qquad (h \le j \le k); \qquad x_i, y_i \in \mathscr{B}. \tag{1.8}$$

Incomplete systems were first studied my Mardzhanishvili ([15], [16]), who gave sufficient conditions for the existence of solutions of the system

$$\sum_{i=1}^{s} x_i^j = N_j \qquad (j \in \mathscr{J}),$$

where $\mathscr{J}$ is an arbitrary finite subset of positive integers. More general systems of Diophantine equations and associated trigonometric sums are treated in [2].

The Vinogradov method [28], when applied to bounding a more general sum

$$\sum_{N<n\le 2N} e(p(n)), \quad p(n) = \alpha_1 n + \cdots + \alpha_k n^k,$$

ultimately depends on having good rational approximations for a subset of the coefficients of $p(n)$, say for $\alpha_i, \alpha_{i+1}, \ldots, \alpha_j$. By applying trivial estimates to sums involving the other coefficients, we may restrict attention to associated mean-values over $\alpha_i, \alpha_{i+1}, \ldots, \alpha_j$ which are equivalent to $J_{s,j,i}(\mathscr{B})$. The core of the argument is given in Lemma 5.1.

When $\mathscr{B} \subseteq [1, P]$, we have a trivial bound

$$J_{s,k,h}(\mathscr{B}) \le s^{h-1} P^{h(h-1)/2} J_{s,k}(P). \tag{1.9}$$

In the application to bounding $S(N, t)$, however, (1.9) gives nothing better than if $J_{s,k,h}(\mathscr{B})$ were replaced by $J_{s,k}(P)$ from the outset. By a more sophisticated method, which is a generalization of the author's work ([5]) on mean values of complete Weyl sums, one can bound $J_{s,k,h}([1, P])$ in terms of $J_{s',k}(P)$ (with $s' < s$), and attain superior bounds for $S(N, t)$. When $\mathscr{B} = \mathscr{A}(P, R)$, the set of numbers $\le P$ with no prime factors exceeding R (R-"smooth" numbers), R is a sufficiently small power of P (depending on k, h, s), and h close to k, Wooley's "efficient differencing" method ([29], [30], [34]) produces even better exponents of P. However, the implied constants coming from the bounds in [34] grow too fast as functions of k, h, s, and thus are inadequate for bounding $S(N, t)$ for the entire range $1 \le \lambda \ll \sqrt{\log N}$. The principal problem is that elements of $\mathscr{A}(P, R)$ may contain a very large number of divisors. We overcome this by taking $\mathscr{B} = \mathscr{C}(P, R)$, the set of integers $\le P$ composed only of prime factors in $(\sqrt{R}, R]$. We thus retain all of the advantages gained by using R-smooth numbers, but now the number of prime factors of each such number is bounded above by $2\frac{\log P}{\log R}$. The next theorem, which will be used for the proof of Theorem 2, is an example of what can be proved.

Theorem 4. *Suppose $k \ge 60$, $0.9k \le h \le k - 2$, $2t \le s \le \lfloor h/2 \rfloor t$, and $P \ge e^{Dk^2}$ where $D \ge 10$. Further assume that*

$$\frac{2}{k^3} < \eta \le \frac{1}{2k}, \quad \frac{18}{k} \le \frac{4\log k}{Dk^2\eta} \le 0.4. \tag{1.10}$$

Then

$$J_{s,k,h}(\mathscr{C}(P, P^\eta)) \le e^C P^{2s - \frac{t}{2}(h+k) + \frac{t(t-1)}{2} + \eta s^2/(2t) + ht\exp\{-s/(ht)\}},$$

where

$$C = \frac{s^2}{t} + \frac{10.5t\log^2 k}{Dk\eta^2} - s\left(\left(\frac{1}{\eta} + h\right)\left(1 - \frac{1}{h}\right)^{s/t} - h\right)\log\left(\frac{1}{10\eta}\right).$$

Sections 2, 3 and 4 are dedicated to proving explicit bounds for $J_{s,k}(P)$ (Theorem 3) and $J_{s,k,h}(\mathscr{C}(P,R))$ (Theorem 4). In §5, we use Vinogradov's method and Theorems 3 and 4 to prove Theorem 2 for large λ. For smaller λ we use older methods (§6), which give better results. This is then applied to the problem of bounding $|\zeta(s)|$ and $|\zeta(s,u)|$ in §7, where Theorem 1 is proved. Lastly, in §8 we discuss the limit of our method, and briefly indicate some ways in which the constant B may be improved a little.

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2. Preliminary Lemmata.

First, we detail some notational conventions. Let $\mathbb{U} = [0,1]$, let $\lfloor x \rfloor$ be the greatest integer $\le x$, let $\lceil x \rceil$ be the smallest integer $\ge x$, write $e(z)$ for $e^{2\pi i z}$ and let $\|x\|$ be the distance from x to the nearest integer. Let $\mathscr{C}(P,R)$ be the set of positive integers $n \le P$, all of whose prime factors are in $(\sqrt{R}, R]$. The functions $\omega(n)$ is the number of distinct prime factors of n, $\Omega(n)$ is the number of prime power divisors of n, $\tau(n)$ is the number of positive divisors of n, and $s_0(n)$ is the product of the distinct primes dividing n (the "square-free kernel" of n). Variables in boldface type always indicate vector quantities with the components using the same letter (e.g. $\boldsymbol{z} = (z_1, z_2, \dots)$).

Lemma 2.1. *If $N > 20$ and $x \ge 2N\log N$, there are at least N primes in the interval $(x, 2x]$. If $0 < \delta \le \frac12$, $\frac{N}{\log N} \ge \frac{6}{\delta}$, $x \ge e^{1.5+1.5/\delta}$ and $x \ge \frac{6}{\delta} N \log N$, then there are at least N primes in the interval $(x, x+\delta x]$.*

Proof. This comes directly from the following inequality due to Rosser and Schoenfeld ([23], Theorems 1 and 2). Let $\pi(x)$ be the number of primes $\le x$. Then for $x > 67$ we have

$$\frac{x}{\log x - 1/2} < \pi(x) < \frac{x}{\log x}\left(1 + \frac{3}{2\log x}\right). \tag{2.1}$$

Thus for $x \ge 1200$, we have $\pi(2x) - \pi(x) \ge 0.735\frac{x}{\log x}$. Taking $x = 2N\log N$ proves the first part of the lemma for $N > 130$. For smaller N we use a short computation. For the second part, from (2.1) we obtain

$$\pi(x+\delta x) - \pi(x) \ge \frac{x(1+\delta)}{\log x}\left(1 + \frac{1/2 - \log(1+\delta)}{\log x}\right) - \frac{x}{\log x} - \frac{3x}{2\log^2 x}.$$

Since $(1+\delta)\log(1+\delta) \le \delta + \frac12\delta^2$, we have

$$\begin{aligned}\pi(x+\delta x) - \pi(x) &\ge \frac{x}{\log x}\left[\delta - \frac{3/2 - (1+\delta)(1/2 - \log(1+\delta))}{\log x}\right] \\ &\ge \frac{x}{\log x}\left[\delta - \frac{1+\delta}{\log x}\right].\end{aligned}$$

Using the lower bounds for x gives

$$\pi(x+\delta x) - \pi(x) \ge \frac{\delta x}{3\log x} \ge \frac{2N\log N}{\log N + \log(\frac{6}{\delta}\log N)} \ge N. \qquad \square$$

Lemma 2.2. *If $0 \le \delta \le \frac{1}{10}$, $u \ge 2-3\delta$ and $R \ge 6^{1/\delta}$, then*

$$|\mathscr{C}(R^u, R)| \ge \frac{\delta^w}{(w+1)!}\frac{R^u}{\log R}, \qquad w = \left\lfloor \frac{u}{1-\delta} \right\rfloor.$$

Proof. Let $N_d(x,R) = |\{n \in \mathscr{C}(x,R) : \Omega(n) \le d\}|$. We show by induction on d that

$$N_d(R^u, R) \ge \frac{\delta^{d-1}}{d!}\frac{R^u}{\log R} \qquad (2-3\delta \le u < d(1-\delta), R \ge 6^{1/\delta}). \tag{2.2}$$

The proof uses another inequality due to Rosser and Schoenfeld ([23], Theorem 5), which states that for some constant B and $x \ge 286$,

$$\left| \sum_{p \le x} \frac{1}{p} - \log\log x - B \right| \le \frac{1}{2\log^2 x}. \tag{2.3}$$

In our applications, $x \ge 6^{1/(2\delta)} \ge 6^5 > 286$. First we establish (2.2) when $d=2$ and $d=3$. Suppose $d=2$ and $2-3\delta \le u < 2-2\delta$. Then $N_2(R^u,R)$ is at least $\frac{1}{2}$ of the number of pairs of primes (p_1,p_2) with $R^{u-1} < p_1 \le R$, $\sqrt{R} < p_2 \le R^u/p_1$. Using $R \ge 6^{1/\delta} \ge 6^{10}$, $R^u/p_1 \ge R^{0.7}$, and (2.1), we have

$$\begin{aligned} N_2(R^u,R) &\ge \frac{1}{2}\sum_{R^{u-1}<p\le R}\left(\pi\left(\frac{R^u}{p}\right) - \pi(\sqrt{R})\right) \\ &\ge \frac{1}{2}\sum_{R^{u-1}<p\le R}\frac{R^u/p}{\log R}\left(1 - 2R^{-0.2}\left(1+\frac{3}{\log R}\right)\right) \\ &\ge \frac{0.46R^u}{\log R}\sum_{R^{u-1}<p\le R}\frac{1}{p}. \end{aligned}$$

By (2.3), the last sum is

$$\ge \log\left(\frac{1}{u-1}\right) - \frac{1}{2\log^2 R}\left(1+\frac{1}{(u-1)^2}\right) \ge \log\left(\frac{1}{1-2\delta}\right) - \frac{\delta^2}{2} \ge 2\delta,$$

and (2.2) follows when $d=2$. Next, let $d=3$. When $2-3\delta \le u < 2-2\delta$, (2.2) follows from the $d=2$ case. If $2-2\delta \le u < 3-3\delta$, define

$$a_1 = \max\left(\frac{u-1}{2}, \frac{1}{2}\right), \qquad a_2 = \min\left(1, \frac{u-1/2-\delta}{2}\right).$$

Then

$$N_3(R^u, R) \ge \frac{1}{6} \sum_{p_1,p_2\in(R^{a_1},R^{a_2}]} \left(\pi\left(\frac{R^u}{p_1p_2}\right) - \pi(\sqrt{R}) \right).$$

For every p_1, p_2,

$$R \ge R^u/p_1p_2 \ge R^{u-2a_2} \ge R^{1/2+\delta}.$$

By (2.1),

$$\begin{aligned}\pi\left(\frac{R^u}{p_1p_2}\right) - \pi(\sqrt{R}) &\ge \frac{R^u/(p_1p_2)}{\log R}\left(1 - 2R^{-\delta}\left(1+\tfrac{3}{\log R}\right)\right)\\ &\ge \frac{0.61R^u}{p_1p_2\log R},\end{aligned}$$

whence

$$N_3(R^u, R) \ge \frac{R^u}{10\log R}\Bigg(\sum_{R^{a_1}<p\le R^{a_2}} \frac{1}{p}\Bigg)^2.$$

By (2.3),

$$\sum_{R^{a_1}<p\le R^{a_2}} \frac{1}{p} \ge \log\left(\frac{a_2}{a_1}\right) - \frac{1}{a_1^2\log^2 R} \ge \log\left(\frac{a_2}{a_1}\right) - 1.25\delta^2.$$

We claim that $\log(a_2/a_1) \ge 1.5\delta$, from which (2.2) follows in the case $d = 3$. Let $I_1 = [2-2\delta, 2)$, $I_2 = [2, 2.5+\delta)$, $I_3 = [2.5+\delta, 3-3\delta)$. Then

$$\log\left(\frac{a_2}{a_1}\right) = \begin{cases} \log(u-1/2-\delta) \ge \log(1.5-3\delta) \ge \log(1+2\delta) \ge 1.5\delta & (u\in I_1)\\ \log\left(\frac{u-1/2-\delta}{u-1}\right) \ge \log\left(\frac{2}{1.5+\delta}\right) \ge \log(1.25) \ge 1.5\delta & (u \in I_2)\\ \log\left(\frac{2}{u-1}\right) \ge \log\left(\frac{2}{2-3\delta}\right) \ge 1.5\delta & (u\in I_3).\end{cases}$$

Next, let $d \ge 3$ and suppose (2.2) holds. When $2 - 3\delta \le u < d(1-\delta)$, (2.2) follows for all larger d as well. Suppose $d(1-\delta) \le u \le (d+1)(1-\delta)$. If $p \in (R^{1-\delta}, R]$, then $R^u/p \in (R^{2-3\delta}, R^{d(1-\delta)}]$, and thus

$$N_d(R^u/p, R) \ge \frac{\delta^{d-1}}{d!}\frac{R^u/p}{\log R}.$$

Summing over primes p, each number pn with n counted by $N_d(R^u/p,R)$ is counted at most $d+1$ times. Hence

$$N_{d+1}(R^u, R) \ge \frac{1}{d+1} \sum_{R^{1-\delta}<p\le R} N_d(R^u/p, R) \ge \frac{\delta^{d-1}}{(d+1)!}\frac{R^u}{\log R} \sum_{R^{1-\delta}<p\le R} \frac{1}{p}.$$

Again using (2.3), the last sum is

$$\ge \log\left(\frac{1}{1-\delta}\right) - \frac{1}{(1-\delta)^2\log^2 R} \ge \delta + \frac{\delta^2}{2} - 0.4\delta^2 > \delta,$$

and (2.2) follows with d replaced by $d+1$. □

Lemma 2.3. *Suppose* $R \ge (2u)^3 \ge 90000$. *Then* $|\mathscr{C}(R^u, R)| \le R^u(2/u)^u$.

Proof. Suppose $\frac{2}{3} \le \beta < 1$ and put $P = R^u$. Then

$$\begin{aligned} |\mathscr{C}(P,R)| &\le P^\beta \sum_{n\in\mathscr{C}(P,R)} n^{-\beta} \le P^\beta \prod_{\sqrt{R}<p\le R} \left(1 + p^{-\beta} + p^{-2\beta} + \cdots\right) \\ &\le P^\beta \exp\left\{ \sum_{\sqrt{R}<p\le R} \frac{1}{p^\beta} + \frac{1}{p^\beta(p^\beta - 1)} \right\} \\ &\le P^\beta \exp\left\{ R^{1-\beta} \sum_{\sqrt{R}<p\le R} \frac{1}{p} + 1.03 \sum_{p>\sqrt{R}} p^{-4/3} \right\}. \end{aligned}$$

Since $\sqrt{R} \ge 300$, by (2.3)

$$\sum_{\sqrt{R}<p\le R} \frac{1}{p} \le \log 2 + \frac{2.5}{\log^2 R} \le 0.713.$$

Also,

$$\sum_{p>\sqrt{R}} p^{-4/3} \le \int_{\sqrt{R}-1}^{\infty} t^{-4/3}\,dt \le 0.45,$$

so that

$$|\mathscr{C}(P,R)| \le P^\beta \exp\{0.713 R^{1-\beta} + 0.47\}.$$

Take $\beta = 1 - \frac{\log(u/0.713)}{\log R} \ge \frac{2}{3}$. Then

$$|\mathscr{C}(P,R)| \le P\exp\{-u\log(u/0.713) + u + 0.47\} = P\left(\frac{0.713e}{u}\right)^u e^{0.47}.$$

Lastly, $u \ge 22$ and thus $(\frac{0.713e}{2})^u e^{0.47} < 1$. $\square$

The next lemma is due to Wooley ([33]), and gives a bound for the number of non-singular solutions of a system of congruences. This greatly generalizes a lemma due to Linnik [14].

Lemma 2.4. *Let* $f_1, \ldots, f_d$ *be polynomials in* $\mathbb{Z}[x_1, \ldots, x_d]$ *with respective degrees* $k_1, \ldots, k_d$, *and write*

$$J(\boldsymbol{f};\boldsymbol{x}) = \det\left(\frac{\partial f_j(\boldsymbol{x})}{\partial x_i}\right)_{1\le i,j\le d}.$$

Also, let p *be a prime number and* s *be a natural number. Then the number,* N, *of solutions of the simultaneous congruences*

$$f_j(x_1, \ldots, x_d) \equiv 0 \pmod{p^s} \quad (1 \le j \le d)$$

with $1 \le x_i \le p^s$ $(1 \le i \le d)$ *and* $(J(\boldsymbol{f};\boldsymbol{x}), p) = 1$, *satisfies* $N \le k_1 \cdots k_d$.

Lastly, we present a general inequality on the number of solutions of "symmetric" systems of equations.

Proposition ZRD (Zero Representation Dominates). *Suppose $f_1, \dots, f_n$ are functions from $\mathbb{Z}^m$ to $\mathbb{Z}$ and $\mathscr{B}$ is a finite subset of $\mathbb{Z}^m$. Let $I(\boldsymbol{f}; \boldsymbol{w}; \mathscr{B})$ be the number of solutions of the simultaneaous Diophantine equations*

$$f_j(\boldsymbol{x}) - f_j(\boldsymbol{y}) = w_j \qquad (1 \le j \le n)$$

with $\boldsymbol{x}, \boldsymbol{y} \in \mathscr{B}$. Then $I(\boldsymbol{f}; \boldsymbol{w}; \mathscr{B}) \le I(\boldsymbol{f}; \mathbf{0}; \mathscr{B})$, where $\mathbf{0} = (0, 0, \dots, 0)$.

Proof. For $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_n)$, let

$$g(\boldsymbol{\alpha}) = \sum_{\boldsymbol{x} \in \mathscr{B}} e(\alpha_1 f_1(\boldsymbol{x}) + \cdots + \alpha_n f_n(\boldsymbol{x})).$$

Then

$$I(\boldsymbol{f}; \boldsymbol{w}; \mathscr{B}) = \int_{\mathbb{U}^n} |g(\boldsymbol{\alpha})|^2 e(-\alpha_1 w_1 - \cdots - \alpha_k w_k)\, d\boldsymbol{\alpha} \le I(\boldsymbol{f}; \mathbf{0}; \mathscr{B}).$$

Alternatively, for $\boldsymbol{v} = (v_1, \cdots, v_n)$, let $n(\boldsymbol{v})$ be the number of solutions of $f_j(\boldsymbol{x}) = v_j$ $(1 \le j \le n)$ with $\boldsymbol{x} \in \mathscr{B}$. By the Cauchy-Schwarz inequality,

$$\begin{aligned} I(\boldsymbol{f}; \boldsymbol{w}; \mathscr{B}) &= \sum_{\substack{\boldsymbol{v}, \boldsymbol{v}' \\ v_j - v_j' = w_j}} n(\boldsymbol{v}) n(\boldsymbol{v}') \\ &\le \Bigg(\sum_{\substack{\boldsymbol{v}, \boldsymbol{v}' \\ v_j - v_j' = w_j}} n(\boldsymbol{v})^2 \Bigg)^{1/2} \Bigg(\sum_{\substack{\boldsymbol{v}, \boldsymbol{v}' \\ v_j - v_j' = w_j}} n(\boldsymbol{v}')^2 \Bigg)^{1/2} = I(\boldsymbol{f}; \mathbf{0}; \mathscr{B}). \qquad \square \end{aligned}$$

3. Vinogradov's Integral: Complete systems

In this section, we derive bounds for $J_{s,k}(P)$ using the iterative methods of Wooley [30], modified using an idea of Arkhipov and Karatsuba [1] (the introduction of the parameter r). It should be noted that using the method of Tyrina [26] when $\frac{4}{9}k^2 \le \Delta(k,s) \le \frac{1}{2}k^2$ gives slightly better values for $\Delta(k,s)$, but only enough to improve the constant B in Theorem 1 by 0.01 or less.

The next definition is slightly different from that given in [30].

Definition. *Suppose* $0 \le d \le k-1$ *and* T *is a positive integer. We say the* k*-tuple of poynomials* $\mathbf{\Psi} = (\Psi_1, \dots, \Psi_k) \in \mathbb{Z}[x]^k$ *is of type* (d,T) *if* Ψ_j *is identically zero for* $j \le d$*, and for some integer* $m \ge 0$*, when* $j > d$*,* Ψ_j *has degree* $j-d$ *with leading coefficient* $\frac{j!}{(j-d)!}2^m T$*.*

Lemma 3.1. *Suppose* $\mathbf{\Psi}$ *is of type* (d,T)*, and* $z_1, \dots, z_{k-d}$ *are integers. Then*

$$J_{k-d}(\boldsymbol{z};\mathbf{\Psi}) := \det\big(\Psi_j'(z_i)\big)_{\substack{1\le i\le k-d\\ d+1\le j\le k}} = (2^m T)^{k-d} \prod_{j=d+1}^{k} \frac{j!}{(j-d-1)!} \prod_{1\le i<j\le k-d} (z_i - z_j),$$

Proof. This follows by elementary row operations. □

The argument will begin with $\Psi_j(z) = z^j$ $(1 \le j \le k)$, which is of type $(0,1)$. At the dth iterative stage $(d \ge 0)$, the system will be transformed from one of type (d,T) to one of type $(d+1,T')$ in two steps. First, for some constant c we will take

$$\Phi_j(z) = \sum_{\ell=0}^{j} \binom{j}{\ell} \Psi_\ell(z) c^{j-\ell},$$

which is also a system of type (d,T). Then, for a constant y we take

$$\Upsilon_j(z) = \Phi_j(z+y) - \Phi_j(z) \qquad (1 \le j \le k),$$

which is of type $(d+1, yT)$.

Fix k and suppose $1 \le r \le k$. If $\mathbf{\Psi} = (\Psi_1, \dots \Psi_k)$ is a system of polynomials, let $K_s(P,Q;\mathbf{\Psi};q)$ be the number of solutions of the simultaneous equations

$$\begin{aligned} &\sum_{i=1}^{k} (\Psi_j(z_i) - \Psi_j(w_i)) + q^j \sum_{i=1}^{s} (x_i^j - y_i^j) = 0 \qquad (1 \le j \le k), \\ &1 \le z_i, w_i \le P; \quad 1 \le x_i, y_i \le Q. \end{aligned} \tag{3.1}$$

Here the inequalities on the variables z_i, w_i, x_i, y_i hold for every i. For prime p, let $L_s(P,Q;\mathbf{\Psi};p,q,r)$ be the number of solutions of

$$\begin{aligned} &\sum_{i=1}^{k} (\Psi_j(z_i) - \Psi_j(w_i)) + (pq)^j \sum_{i=1}^{s} (u_i^j - v_i^j) = 0 \qquad (1 \le j \le k), \\ &1 \le z_i, w_i \le P;\ z_i \equiv w_i \pmod{p^r};\ 1 \le u_i, v_i \le Q. \end{aligned} \tag{3.2}$$

Define the exponential sums

$$f(\boldsymbol{\alpha}) = f(\boldsymbol{\alpha}; Q; q) = \sum_{x \le Q} e(\alpha_1 qx + \cdots + \alpha_k q^k x^k),$$
$$F(\boldsymbol{\alpha}) = F(\boldsymbol{\alpha}; P; \boldsymbol{\Psi}) = \sum_{x \le P} e(\alpha_1 \Psi_1(x) + \cdots + \alpha_k \Psi_k(x)).$$

Then

$$K_s(P, Q; \boldsymbol{\Psi}; q) = \int_{\mathbb{U}^k} |F(\boldsymbol{\alpha})^{2k} f(\boldsymbol{\alpha})^{2s}| \, d\boldsymbol{\alpha}.$$

The next result relates K_s and L_s, and is a generalization of the "fundamental lemma" of Wooley ([30], Lemma 3.1).

Lemma 3.2. *Suppose k, r, d and s are integers with*

$$k \ge 4, \; 2 \le r \le k; \; 0 \le d \le r-1; \; s \ge d+1.$$

Let M, P and Q be real numbers with

$$P^{\frac{1}{k+1}} \le M \le P^{\frac{1}{r}}; \quad 32s^2 M < Q \le P; \quad M \ge k.$$

Suppose q is a positive integer and $\boldsymbol{\Psi}$ is a system of polynomials of type (d, T) with $T \le P^d$. Denote by $\mathscr{P}$ the set of the k^3 smallest primes $> M$, and suppose $\mathscr{P} \subset (M, 2M]$. Then there is a system of polynomials $\boldsymbol{\Phi}$ of type (d, T) and a prime $p \in \mathscr{P}$ such that

$$K_s(P, Q; \boldsymbol{\Psi}; q) \le 4k^3 k! p^{2s + \frac{1}{2}(r^2 - r + d^2 - d)} L_s(P, \tfrac{Q}{p}; \boldsymbol{\Phi}; p, q, r).$$

Proof. Let W be the set of systems of polynomials of type (d, T) with $T \le P^d$. Since $K_s(P, Q; \boldsymbol{\Psi}; q) \le P^{2k} Q^{2s}$ trivially, there is a system $\boldsymbol{\Psi}_0 \in W$ so that

$$K_s(P, Q; \boldsymbol{\Psi}_0; q) = \max_{\boldsymbol{\Psi} \in W} K_s(P, Q; \boldsymbol{\Psi}; q).$$

We therefore assume without loss of generality that $\boldsymbol{\Psi} = \boldsymbol{\Psi}_0$. For brevity, write K for $K_s(P, Q; \boldsymbol{\Psi}; q)$. We divide the solutions of (3.1) into two classes: S_2 is the number of solutions with $z_i = z_j$ or $w_i = w_j$ for some $i \ne j$; S_1 is the number of remaining solutions. Clearly $K \le 2\max(S_1, S_2)$. Suppose first that $S_2 \ge S_1$. By Hölder's inequality,

$$\begin{aligned} K &\le 2S_2 \le 4\binom{k}{2} \int_{\mathbb{U}^k} |F(\boldsymbol{\alpha})^{2k-2} F(2\boldsymbol{\alpha}) f(\boldsymbol{\alpha})^{2s}| \, d\boldsymbol{\alpha} \\ &< 2k^2 \left(\int_{\mathbb{U}^k} |F(\boldsymbol{\alpha})^{2k} f(\boldsymbol{\alpha})^{2s}| \, d\boldsymbol{\alpha} \right)^{1-\frac{1}{k}} \left(\int_{\mathbb{U}^k} |f(\boldsymbol{\alpha})|^{2s} \, d\boldsymbol{\alpha} \right)^{\frac{1}{2k}} \left(\int_{\mathbb{U}^k} |F(2\boldsymbol{\alpha})^{2k} f(\boldsymbol{\alpha})^{2s}| \, d\boldsymbol{\alpha} \right)^{\frac{1}{2k}} \\ &= 2k^2 K^{1-1/k} (J_{s,k}(Q))^{1/2k} K_s(P, Q; 2\boldsymbol{\Psi}; q) \\ &\le 2k^2 K^{1-1/2k} (J_{s,k}(Q))^{1/2k}. \end{aligned}$$

∎

Here $2\boldsymbol{\Psi} = (2\Psi_1(z), \dots, 2\Psi_k(z))$ is also of type (d, T), which justifies the last inequality above. This is the reason for the introduction of the parameter m in the definition of a system of polynomials of type (d, T). Therefore $K \le (2k^2)^{2k} J_{s,k}(Q)$. On the other hand, counting the solutions of (3.1) with $z_i = w_i$ for each i produces the lower bound $K \ge (P-1)^k J_{s,k}(Q)$. The hypothesis $\mathscr{P} \subset (M, 2M]$ gives $M \ge k^3 - 1$ and so $P - 1 \ge (k^3-1)^2 - 1 > 4k^4$. We have a contradiction, therefore $K \le 2S_1$. To bound S_1, we follow the procedure from Wooley [30]. Consider a solution of (3.1) counted by S_1. By Lemma 3.1, for some integer $m \ge 0$ we have

$$J_{k-d}(\boldsymbol{z}; \boldsymbol{\Psi}) J_{k-d}(\boldsymbol{w}; \boldsymbol{\Psi}) = (2^m T)^{2k-2d} \prod_{j=d+1}^{k} \left(\tfrac{j!}{(j-d-1)!} \right)^2 \prod_{1 \le i < j \le k-d} (z_i - z_j)(w_i - w_j) \neq 0.$$

By hypothesis, if $p \in \mathscr{P}$ then $p > M \ge k$. Also,

$$\left| T \prod_{1 \le i < j \le k-d} (z_i - z_j)(w_i - w_j) \right| < P^{d + (k-d)(k-d-1)} \le P^{k^2 - k} < \prod_{p \in \mathscr{P}} p.$$

Thus, for each solution counted by S_1, there is some $p \in \mathscr{P}$ which does not divide $J_{k-d}(\boldsymbol{z}; \boldsymbol{\Psi}) J_{k-d}(\boldsymbol{w}; \boldsymbol{\Psi})$. Hence

$$K \le 2k^3 \max_{p \in \mathscr{P}} S_3(p), \tag{3.3}$$

where $S_3(p)$ is the number of solutions of (3.1) with $(p, J_{k-d}(\boldsymbol{z}; \boldsymbol{\Psi}) J_{k-d}(\boldsymbol{w}; \boldsymbol{\Psi})) = 1$. With p fixed, let

$$g(\boldsymbol{\alpha}; b) = \sum_{\substack{x \le Q \\ x \equiv b \pmod p}} e(\alpha_1 q x + \cdots + \alpha_k q^k x^k),$$

$$\widetilde{F}(\boldsymbol{\alpha}) = \sum_{\substack{z_1, \dots, z_k \\ (J_{k-d}(\boldsymbol{z}; \boldsymbol{\Psi}), p) = 1}} e\left(\sum_{j=1}^{k} \alpha_j (\Psi_j(z_1) + \cdots + \Psi_j(z_k)) \right).$$

Since $\boldsymbol{\Psi}$ is of type (d, T), for any solution of (3.1) we have

$$\sum_{i=1}^{s} (x_i^j - y_i^j) = 0 \qquad (1 \le j \le d).$$

Let $\mathscr{B}_s(\boldsymbol{w})$ denote the set of solutions (with $0 \le c_i \le p-1$ for each i) of the system of congruences

$$\sum_{i=1}^{s} c_i^j \equiv w_j \pmod p \qquad (1 \le j \le d).$$

Consequently,

$$S_3(p) \le \int_{\mathbb{U}^k} |\widetilde{F}(\boldsymbol{\alpha})|^2 \sum_{\substack{\boldsymbol{w} \\ 1\le w_j \le p}} |U(\boldsymbol{\alpha};\boldsymbol{w})|^2 \, d\boldsymbol{\alpha},$$

where

$$U(\boldsymbol{\alpha};\boldsymbol{w}) = \sum_{\boldsymbol{c}\in\mathscr{B}_s(\boldsymbol{w})} g(\boldsymbol{\alpha};c_1)\cdots g(\boldsymbol{\alpha};c_s).$$

By first fixing $c_{d+1},\dots,c_s$, we have $|\mathscr{B}_s(\boldsymbol{w})| \le p^{s-d}\max_{\boldsymbol{v}} |\mathscr{B}_d(\boldsymbol{v})|$. Suppose $\boldsymbol{c}$ and $\boldsymbol{c}'$ are two solutions counted in $\mathscr{B}_d(\boldsymbol{v})$. Let $q(t)=(t-c_1)\cdots(t-c_d)$. By Newton's formulas connecting the sums of the powers of the roots of a polynomial with its coefficients, $q(t)\equiv (t-c_1')\cdots(t-c_d') \pmod p$. Thus, $\boldsymbol{c}'$ is a permutation of $\boldsymbol{c}$, whence $|\mathscr{B}_d(\boldsymbol{v})|\le d!$ and

$$|\mathscr{B}_s(\boldsymbol{w})| \le d! p^{s-d}.$$

By the Cauchy-Schwarz inequality , followed by an application of the arithmetic mean-geometric mean inequality, we have

$$\begin{aligned} |U(\boldsymbol{\alpha};\boldsymbol{w})|^2 &\le |\mathscr{B}_s(\boldsymbol{w})| \sum_{\boldsymbol{c}\in\mathscr{B}_s(\boldsymbol{w})} |g(\boldsymbol{\alpha};c_1)\cdots g(\boldsymbol{\alpha};c_s)|^2 \\ &\le \frac{d!}{s} p^{s-d} \sum_{\boldsymbol{c}\in\mathscr{B}_s(\boldsymbol{w})} \sum_{i=1}^{s} |g(\boldsymbol{\alpha};c_i)|^{2s}. \end{aligned}$$

We then have

$$\begin{aligned} S_3(p) &\le d! p^{s-d} \sum_{\boldsymbol{c}} \max_{1\le i\le s} \int_{\mathbb{U}^k} |\widetilde{F}(\boldsymbol{\alpha})|^2 |g(\boldsymbol{\alpha};c_i)|^{2s}\, d\boldsymbol{\alpha} \\ &\le d! p^{2s-d} \max_{0\le c\le p-1} S_4(c,p), \end{aligned} \tag{3.4}$$

where

$$S_4(c,p) = \int_{\mathbb{U}^k} |\widetilde{F}(\boldsymbol{\alpha})^2 g(\boldsymbol{\alpha};c)^{2s}|\, d\boldsymbol{\alpha}$$

is the number of solutions of

$$\begin{gathered} \sum_{i=1}^{k} (\Psi_j(z_i)-\Psi_j(w_i)) + q^j \sum_{i=1}^{s} ((pu_i-c)^j - (pv_i-c)^j) = 0 \quad (1\le j\le k), \\ 1\le z_i,w_i\le P; \quad (p, J_{k-d}(\boldsymbol{z};\boldsymbol{\Psi})J_{k-d}(\boldsymbol{w};\boldsymbol{\Psi}))=1; \quad 1\le u_i,v_i\le (Q+c)/p. \end{gathered} \tag{3.5}$$

Let $S_5(c,p)$ denote the number of solutions of (3.5) with $u_i > Q/p$ or $v_i > Q/p$ for some i, and let $S_6(c,p)$ denote the number remaining solutions. Suppose first that $S_5(c,p)\ge S_6(c,p)$. By Hölder's inequality,

$$\begin{aligned} S_4(c,p) \le 2S_5(c,p) &\le 4s \int_{\mathbb{U}^k} |\widetilde{F}(\boldsymbol{\alpha})^2 g(\boldsymbol{\alpha};c)^{2s-1}|\, d\boldsymbol{\alpha} \\ &\le 4s \left(\int_{\mathbb{U}^k} |\widetilde{F}(\boldsymbol{\alpha})^2 g(\boldsymbol{\alpha};c)^{2s}|\, d\boldsymbol{\alpha} \right)^{1-\frac{1}{2s}} \left(\int_{\mathbb{U}^k} |\widetilde{F}(\boldsymbol{\alpha})|^2\, d\boldsymbol{\alpha} \right)^{\frac{1}{2s}} \\ &= 4s\,(S_4(c,p))^{1-\frac{1}{2s}} \left(\int_{\mathbb{U}^k} |\widetilde{F}(\boldsymbol{\alpha})|^2\, d\boldsymbol{\alpha} \right)^{\frac{1}{2s}}. \end{aligned}$$

Therefore,

$$S_4(c,p) \le (4s)^{2s} \int_{\mathbb{U}^k} |\widetilde{F}(\boldsymbol{\alpha})|^2 \, d\boldsymbol{\alpha}.$$

Note that $\lfloor (Q+c)/p \rfloor > Q/p$ in this case. Thus, counting only the solutions of (3.5) with $u_i = v_i$ for every i gives

$$S_4(c,p) \ge (Q/p)^s \int_{\mathbb{U}^k} |\widetilde{F}(\boldsymbol{\alpha})|^2 \, d\boldsymbol{\alpha}.$$

By our assumed lower bound on Q, this is impossible. Therefore, $S_4(c,p) \le 2S_6(c,p)$. By the binomial theorem,

$$(py)^j = \sum_{\ell=0}^{j} \binom{j}{\ell} (py-c)^\ell c^{j-\ell}.$$

Thus, $S_6(c,p)$ is the number of solutions of

$$\begin{gathered} \sum_{i=1}^{k} (\Phi_j(z_i) - \Phi_j(w_i)) + (pq)^j \sum_{i=1}^{s} (u_i^j - v_i^j) = 0 \qquad (1 \le j \le k), \\ 1 \le z_i, w_i \le P; \quad (p, J_{k-d}(\boldsymbol{z}; \boldsymbol{\Psi}) J_{k-d}(\boldsymbol{w}; \boldsymbol{\Psi})) = 1; \quad 1 \le u_i, v_i \le Q/p, \end{gathered} \tag{3.6}$$

where, for $1 \le j \le k$,

$$\Phi_j(z) = \sum_{\ell=0}^{j} \binom{j}{\ell} \Psi_\ell(z) c^{j-\ell}.$$

The leading coefficients of Φ_j and Ψ_j are equal, hence $\boldsymbol{\Phi}$ is also of type (d,T) (with the same value of m). By Lemma 3.1, $J_{k-d}(\boldsymbol{z}; \boldsymbol{\Psi}) = J_{k-d}(\boldsymbol{z}; \boldsymbol{\Phi})$, so $(p, J_{k-d}(\boldsymbol{z}; \boldsymbol{\Phi}) J_{k-d}(\boldsymbol{w}; \boldsymbol{\Phi})) = 1$ in (3.6).

Lastly, we introduce the congruence condition on z_i, w_i. By (3.6),

$$\sum_{i=1}^{k} (\Phi_j(z_i) - \Phi_j(w_i)) \equiv 0 \pmod{p^j} \qquad (1 \le j \le k).$$

We shall only work with the congruences corresponding to $d+1 \le j \le k$, since the left side of the above congruence is identically zero when $j \le d$. Let $\mathscr{B}^*(\boldsymbol{m})$ be the set of $\boldsymbol{z}$ with $1 \le z_i \le p^r$ for each i, $(J_{k-d}(\boldsymbol{z}; \boldsymbol{\Phi}), p) = 1$ and

$$\sum_{i=1}^{k} \Phi_j(z_i) \equiv m_j \pmod{p^{\min(j,r)}} \qquad (d+1 \le j \le k).$$

By hypothesis, $d+1 \le r$. To bound $|\mathscr{B}^*(\boldsymbol{m})|$, first fix $z_{k-d+1}, \ldots, z_k$ (there are p^{rd} such choices). For each j, there are $p^{\max(0,r-j)}$ possibilities for m_j modulo p^r, and with the m_j fixed modulo p^r, Lemma 2.4 implies that there are at most $(k-d)!$ solutions $z_1, \dots, z_{k-d}$ modulo p^r. Therefore,

$$|\mathscr{B}^*(\boldsymbol{m})| \le (k-d)! p^{\frac12 (r-d-1)(r-d)+rd}.$$

Define

$$H(\boldsymbol{\alpha};\boldsymbol{z}) = \sum_{\substack{\boldsymbol{w} \\ 1\le w_i\le P \\ w_i\equiv z_i \pmod{p^r}}} e\left(\sum_{j=1}^k \alpha_j (\Phi_j(w_1)+\cdots+\Phi_j(w_k)) \right).$$

Then, by the Cauchy-Schwarz inequality ,

$$\begin{aligned} S_6(c,p) &\le \int_{\mathbb{U}^k} \sum_{\boldsymbol{m}} \bigg| \sum_{\boldsymbol{z}\in\mathscr{B}^*(\boldsymbol{m})} H(\boldsymbol{\alpha};\boldsymbol{z}) \bigg|^2 |f(\boldsymbol{\alpha};Q/p;pq)|^{2s}\, d\boldsymbol{\alpha} \\ &\le \sum_{\boldsymbol{m}} |\mathscr{B}^*(\boldsymbol{m})| \int_{\mathbb{U}^k} \sum_{\boldsymbol{z}\in\mathscr{B}^*(\boldsymbol{m})} |H(\boldsymbol{\alpha};\boldsymbol{z})|^2 |f(\boldsymbol{\alpha};Q/p;pq)|^{2s}\, d\boldsymbol{\alpha} \\ &\le (k-d)! p^{\frac12 (r-d-1)(r-d)+rd} L_s(P,Q/p;\boldsymbol{\Phi};p,q,r). \end{aligned}$$

By (3.4) and the inequality $d!(k-d)! \le k!$,

$$S_3(p) \le 2k! p^{2s-d+\frac12 (r-d-1)(r-d)+rd} L_s(P,Q/p;\boldsymbol{\Phi};p,q,r). \tag{3.7}$$

The lemma now follows from (3.3). □

Lemma 3.3. *Suppose that $s\ge d$, $k\ge r\ge 2$, $d\le k-2$, $q\ge 1$, p is a prime and $\boldsymbol{\Phi}$ is a system of polynomials of type (d,T). Then there is a system of polynomials $\boldsymbol{\Upsilon}$ of type $(d+1,T')$ with $T\le T'\le PT$ such that*

$$L_s(P;Q;\boldsymbol{\Phi};p,q,r) \le (2P)^k \max\left[k^k J_{s,k}(Q), 2p^{-rk} \left\{ J_{s,k}(Q) K_s(P,Q;\boldsymbol{\Upsilon};pq) \right\}^{1/2} \right].$$

Proof. For short, write L for $L_s(P;Q;\boldsymbol{\Phi};p,q,r)$. Then $L \le 2\max(U_0,U_1)$, where U_0 is the number of solutions of (3.2) with $w_i=z_i$ for some i, and U_1 is the number of solutions of (3.2) with $w_i\ne z_i$ for every i. First write $f(\boldsymbol{\alpha})$ for $f(\boldsymbol{\alpha};Q;pq)$ and

$$I(\boldsymbol{\alpha}) = \sum_{1\le c\le P} \bigg| \sum_{\substack{1\le w\le P \\ w\equiv c \pmod{p^r}}} e(\alpha_1\Phi_1(w)+\cdots+\alpha_k\Phi_k(w)) \bigg|^2,$$

so that

$$L = \int_{\mathbb{U}^k} I(\boldsymbol{\alpha})^k |f(\boldsymbol{\alpha})|^{2s}\, d\boldsymbol{\alpha}.$$

Suppose first that $U_0 \geq U_1$. By Hölder's inequality,

$$\begin{aligned} L \leq 2U_0 &\leq 2kP \int_{\mathbb{U}^k} I(\boldsymbol{\alpha})^{k-1} |f(\boldsymbol{\alpha})|^{2s}\, d\boldsymbol{\alpha} \\ &\leq 2kP \left(\int_{\mathbb{U}^k} |I(\boldsymbol{\alpha})^k f(\boldsymbol{\alpha})^{2s}|\, d\boldsymbol{\alpha} \right)^{1-1/k} \left(\int_{\mathbb{U}^k} |f(\boldsymbol{\alpha})|^{2s}\, d\boldsymbol{\alpha} \right)^{1/k} \\ &= 2kPL^{1-1/k} J_{s,k}(Q)^{1/k}, \end{aligned}$$

and the lemma follows in this case. If $U_1 \geq U_0$, for each i we may write $w_i = z_i + h_i p^r$, where $1 \leq |h_i| \leq P/p^r$. We may assume that $P/p^r \geq 1$, else $U_1 = 0$. Let

$$g(\boldsymbol{\alpha}; h) = \sum_{1 \leq z \leq P} e\left(\sum_{j=1}^{k} \alpha_j (\Phi_j(z + hp^r) - \Phi_j(z)) \right).$$

There are 2^k choices for the signs of $w_i - z_i$ $(1 \leq i \leq k)$, so

$$L \leq 2 \sum_{\substack{\eta_1, \dots, \eta_k \\ \eta_i \in \{-1,+1\}}} \int_{\mathbb{U}^k} \sum_{\substack{\boldsymbol{h} \\ 1 \leq h_i \leq P/p^r}} g(\eta_1 \boldsymbol{\alpha}; h_1) \cdots g(\eta_k \boldsymbol{\alpha}; h_k) |f(\boldsymbol{\alpha})|^{2s}\, d\boldsymbol{\alpha}.$$

Since $|g(\boldsymbol{\alpha}; h)| = |g(-\boldsymbol{\alpha}; h)|$,

$$\sum_{\substack{\boldsymbol{h} \\ 1 \leq h_i \leq P/p^r}} |g(\eta_1 \boldsymbol{\alpha}; h_1) \cdots g(\eta_k \boldsymbol{\alpha}; h_k)| \leq (P/p^r)^k \max_{1 \leq h \leq P/p^r} |g(\boldsymbol{\alpha}; h)|^k.$$

Then, by the Cauchy-Schwarz inequality ,

$$\begin{aligned} L &\leq 2^{k+1} (P/p^r)^k \max_{1 \leq h \leq P/p^r} \int_{\mathbb{U}^k} |g(\boldsymbol{\alpha}; h)|^k |f(\boldsymbol{\alpha})|^{2s}\, d\boldsymbol{\alpha} \\ &\leq 2^{k+1} (P/p^r)^k \max_{1 \leq h \leq P/p^r} \left(\int_{\mathbb{U}^k} |g(\boldsymbol{\alpha}; h)|^{2k} |f(\boldsymbol{\alpha})|^{2s}\, d\boldsymbol{\alpha} \right)^{1/2} \left(\int_{\mathbb{U}^k} |f(\boldsymbol{\alpha})|^{2s}\, d\boldsymbol{\alpha} \right)^{1/2} \\ &= 2^{k+1} (P/p^r)^k \max_{1 \leq h \leq P/p^r} \left(K_s(P, Q; \boldsymbol{\Upsilon}; pq) J_{s,k}(Q) \right)^{1/2}, \end{aligned}$$

where $\Upsilon_j(z) = \Phi_j(z + hp^r) - \Phi_j(z)$ for $j \geq d+2$ and $\Upsilon_j(z) \equiv 0$ for $j \leq d+1$. For some integer $m \geq 0$ and $j \geq d+2$, Υ_j has degree $j - d - 1$ and leading coefficient $\frac{j!}{(j-d-1)!} hp^r 2^m T$, thus the system $\boldsymbol{\Upsilon}$ is of type $(d+1, Thp^r)$. □

Next, we iterate Lemmas 3.2 and 3.3 to produce a bound for $J_{s+k,k}(P)$ in terms of the bounds for $J_{s,k}(Q)$.

Lemma 3.4. *Suppose $k \ge 26$, $4 \le r \le k$, $k \le s \le k^3$ and*

$$J_{s,k}(Q) \le CQ^{2s-\frac12 k(k+1)+\Delta} \qquad (Q \ge 1).$$

Let j be an integer satisfying

$$2 \le j \le \tfrac{9r}{10}, \quad (j-1)(j-2) \le 2\Delta - (k-r)(k-r+1). \tag{3.8}$$

Define

$$\phi_j = \frac{1}{r}, \quad \phi_J = \frac{1}{2r} + \frac{k^2+k+r^2-r+J^2-J-2\Delta}{4kr}\phi_{J+1} \quad (1 \le J \le j-1),$$

and suppose r and j are chosen so that $\phi_i \ge \frac{1}{k+1}$ for every i. Suppose

$$\frac{1}{3\log k} \le \omega \le \frac12, \quad \eta = 1+\omega, \quad V = \max\left(e^{1.5+1.5/\omega}, \frac{18}{\omega}k^3\log k\right).$$

If $P \ge V^{k+1}$, then

$$J_{s+k,k}(P) \le k^{3k}\eta^{4s+k^2}CP^{2(s+k)-\frac12 k(k+1)+\Delta'},$$

where $\Delta' = \Delta(1-\phi_1) - k + \frac{\phi_1}{2}(k^2+k+r^2-r)$.

Proof. Let $Q_0 = P$ and for $1 \le i \le j$ define

$$M_i = P^{\phi_i}, \qquad Q_i = P^{1-(\phi_1+\cdots+\phi_i)}.$$

Let $\mathscr{P}_i$ be the set of k^3 smallest primes $> M_i$. By hypothesis, $M_i \ge V$, and by the definition of η and V, Lemma 2.1 implies that $\mathscr{P}_i \subset (M_i, \eta M_i]$. By (3.8), $\phi_i \le \frac1r$ for each i, and for $i \le j-1$

$$\begin{aligned} Q_i \ge Q_{j-1} &\ge P^{1-(j-1)/r} \ge P^{1/10+1/r} \\ &\ge V^{k/10}P^{\phi_{i+1}} > k^8 P^{\phi_{i+1}} > 32s^2P^{\phi_{i+1}}. \end{aligned} \tag{3.9}$$

Let $\lambda = 2s - \frac12 k(k+1)+\Delta$. We shall show by induction on J that for every system $\mathbf{\Phi}$ of type (J,T) with $1 \le T \le P^J$, every prime $p \in \mathscr{P}_{J+1}$ and every positive integer q,

$$L_s(P, Q_{J+1}; \mathbf{\Phi}; p, q, r) \le E_J C P^k Q_{J+1}^{\lambda}, \tag{3.10}$$

where

$$E_{j-1} = 1, \quad E_{J-1} = k^k \eta^{s+\frac14(k^2-k+J^2-J)}E_J^{1/2} \ (1 \le J \le j-1).$$

First, when $J = j-1$, we have $p^r > M_{j-1}^r \ge P$, so that in (3.2), $w_i = z_i$ for every i. This gives

$$L_s(P, Q_j; \mathbf{\Phi}; p, q, r) \le P^k J_{s,k}(Q_j),$$

which gives (3.10) for $J = j-1$. Now suppose $1 \le J \le j-1$ and (3.10) holds. Let $\mathbf{\Psi}$ be a system of polynomials of type (J,T) with $1 \le T \le P^J$, and let q' be any positive integer. By (3.9), (3.10) and the fact that $L_s(P,Q;\mathbf{\Phi};p,q,r)$ is a non-decreasing function of Q, we find from Lemma 3.2 that

$$K_s(P,Q_J;\mathbf{\Psi};q') \le 4k^3k!(\eta M_{J+1})^{2s+\frac{1}{2}(r^2-r+J^2-J)}E_J C P^k Q_{J+1}^{\lambda}.$$

By Lemma 3.3, for every system of polynomials $\mathbf{\Phi}$ of type $(J-1,T)$ with $1 \le T \le P^{J-1}$, prime $p \in \mathscr{P}_J$ and integer q, there is a system $\mathbf{\Psi}$ of polynomials of type (J,T') with $T' \le P^J$ such that

$$\begin{aligned} L_s(P,Q_J;\mathbf{\Phi};p,q,r) &\le (2P)^k \max\left[k^k C Q_J^{\lambda}, 2P^{-kr\phi_J}\left(CQ_J^{\lambda}K_s(P,Q_J;\mathbf{\Psi};pq)\right)^{\frac{1}{2}}\right] \\ &\le CQ_J^{\lambda}(2P)^k \max\left[k^k, 4(k^3k!)^{\frac{1}{2}}E_J^{\frac{1}{2}}P^{\frac{k}{2}-kr\phi_J}M_{J+1}^{-\frac{\lambda}{2}}(\eta M_{J+1})^{s+\frac{1}{4}(r^2-r+J^2-J)}\right]. \end{aligned}$$

By the definition of ϕ_i,

$$\frac{k}{2} - kr\phi_J + \frac{1}{2}\left(\frac{k(k+1)}{2} - \Delta + \frac{1}{2}(r^2-r+J^2-J)\right)\phi_{J+1} = 0,$$

i.e.,

$$P^{k/2-kr\phi_J}M_{J+1}^{s-\lambda/2+\frac{1}{4}(r^2-r+J^2-J)} = 1.$$

Since $r \le k$ and $4(k^3k!)^{1/2} \le 2^{-k}k^k$ for $k \ge 8$, this implies

$$L_s(P,Q_J;\mathbf{\Phi};p,q,r) \le CQ_J^{\lambda}(kP)^k \max\left(2^k, E_J^{1/2}\eta^{s+\frac{1}{4}(k^2-k+J^2-J)}\right).$$

Next, $E_J \ge 1$ and

$$\eta^{s+\frac{1}{4}(k^2-k)} \ge \left(\left(1+\frac{1}{3\log k}\right)^{\frac{k+3}{4}}\right)^k \ge 2^k \quad (k \ge 26).$$

Therefore, by the definition of E_{J-1},

$$L_s(P,Q_J;\mathbf{\Phi};p,q,r) \le CE_{J-1}P^kQ_J^{\lambda},$$

i.e., (3.10) follows with J replaced by $J-1$. Finally, taking (3.10) with $J=0$ and applying Lemma 3.2 with $\Psi_j(x) = x^j$ for each j gives

$$\begin{aligned} K_s(P,P;\mathbf{\Psi};1) &\le 4k^3k!(\eta M_1)^{2s+\frac{1}{2}(r^2-r)}E_0CP^kQ_1^{\lambda} \\ &\le CP^{\lambda+k}4k^3k!\eta^{2s+\frac{1}{2}(k^2-k)}E_0M_1^{\frac{1}{2}(k^2+k+r^2-r)-\Delta}. \end{aligned}$$

From the definition of E_J, we have

$$\begin{aligned} E_0 &= \prod_{J=1}^{j-1} \left(\frac{E_{J-1}}{\sqrt{E_J}} \right)^{2^{1-J}} E_{j-1}^{2^{1-j}} \\ &\le \prod_{J=1}^{\infty} \left(k^k \eta^{s+\frac14 (k^2-k+J^2-J)} \right)^{2^{1-J}} \\ &= k^{2k} \eta^{2s+\frac12 k^2 - \frac12 k + 2}. \end{aligned}$$

Lastly, $4k^3 k! \le k^k$ for $k \ge 11$. Therefore

$$J_{s+k,k}(P) = K_s(P,P;\mathbf{\Psi};1) \le k^{3k} \eta^{4s+k^2} C P^{2(s+k)-\frac12 k(k+1)+\Delta'}. \qquad \square$$

For a given k, r, Δ, we let $\delta_0(k,r,\Delta)$ be the value of Δ' coming from Lemma 3.4, where we take j maximal satisfying (3.8). The optimal value of r is about $\sqrt{k^2+k-2\Delta}$, but leads to very messy analysis. Making the choice $r \approx k(1-\Delta/k^2)$ simplifies matters and ultimately increases the value of B in Theorem 1 by only about 0.0074.

Lemma 3.5. *Let $k \ge 26$ and let ω, η and V be as in Lemma 3.4. Let $\Delta_1 = \frac12 k^2(1-1/k)$ and for $n \ge 1$, let r_n be an integer in $[4,k]$ satisfying*

$$\phi^*(k, r_n, \Delta_n) := \frac{2k}{2r_n k + 2\Delta_n - (k-r_n)(k-r_n+1)} \ge \frac{1}{k+1}, \tag{3.11}$$

then set $\Delta_{n+1} = \delta_0(k, r_n, \Delta_n)$. If $n \le k^2$, then

$$J_{nk,k}(P) \le C_n P^{2nk - \frac12 k(k+1) + \Delta_n} \qquad (P \ge 1),$$

where $C_1 = k!$ and for $n \ge 2$

$$C_n = C_{n-1} \max \left[k^{3k} \eta^{4k(n-1)+k^2}, V^{(k+1)(\Delta_{n-1}-\Delta_n)} \right].$$

Proof. Defining ϕ_i as in Lemma 3.4, we must ensure that $\phi_i \ge \frac{1}{k+1}$ for each i. To this end, let $r = r_n$, $\Delta = \Delta_n$, $\phi^* = \phi^*(k,r,\Delta)$ and $y = 2\Delta - (k-r)(k-r+1)$. For $i \ge 1$ let $\theta_i = \phi_i - \phi^*$. By (3.8), $y - (j-1)(j-2) \ge 0$, so $\theta_j = 1/r - \phi^* \ge 0$. Also,

$$\theta_J = \frac{\theta_{J+1}}{4kr}(2rk + J^2 - J - y) + \frac{J^2 - J}{4kr}\phi^* \quad (1 \le J \le j-1).$$

Since $2\Delta \le k^2 - k$, $0 \le 2rk + J^2 - J - y \le 2rk$. It follows that for $J \le j-1$,

$$0 \le \theta_J \le \frac{\theta_{J+1}}{2} + \frac{J^2 - J}{4kr}\phi^*. \tag{3.12}$$

Thus, (3.11) and (3.12) imply that $\phi_i \ge \phi^* \ge \frac{1}{k+1}$ for every i. We now proceed by induction, noting that the lemma holds with $n = 1$ by the inequality $J_{k,k}(P) \le k!P^k$. Assume now that $m \ge 2$ and the lemma holds for $n \le m-1$. By Lemma 3.4,

$$J_{mk,k}(P) \le C_{m-1} k^{3k} \eta^{4k(m-1)+k^2} P^{2mk - \frac{1}{2}k(k+1)+\Delta_m} \qquad (P \ge V^{k+1}).$$

For $P < V^{k+1}$, we have trivially

$$\begin{aligned} J_{mk,k}(P) &\le P^{2k} J_{(m-1)k,k}(P) \le C_{m-1} P^{2mk-\frac{1}{2}k(k+1)+\Delta_{m-1}} \\ &\le C_{m-1} V^{(k+1)(\Delta_{m-1}-\Delta_m)} P^{2mk-\frac{1}{2}k(k+1)+\Delta_m}. \end{aligned}$$

This completes the proof. □

For a particular choice of $r_1, r_2, \cdots$, the next lemma gives clean upper bounds on Δ_n and C_n for large k.

Lemma 3.6. *Suppose that $k \ge 1000$. For*

$$2k \le n \le \frac{k}{2}\left(\frac{1}{2} + \log\left(\frac{3k}{8}\right)\right) + 1,$$

we have

$$J_{nk,k}(P) \le C_n P^{2nk-\frac{1}{2}k(k+1)+\Delta_n} \qquad (P \ge 1),$$

where

$$\begin{aligned} \Delta_n &\le \frac{3}{8} k^2 e^{1/2-2n/k+1.69/k}, \\ C_n &\le k^{2.055k^3-5.91k^2+3nk} 1.06^{nk^2+2k(n^2-n)-9.7278k^3}. \end{aligned}$$

Proof. We shall take $r_n = \lfloor k - \Delta_n/k + 1 \rfloor$ in Lemma 3.5. For each n write $\delta_n = \Delta_n/k^2$. Fix $n \ge 2$ and write $\delta = \delta_{n-1}$, $\delta' = \delta_n$, $\Delta = \Delta_{n-1}$, $\Delta' = \Delta_n$, $r = r_{n-1}$. If $\Delta_{n-1} \le k$, the upper bound for Δ_n in the lemma follows from the upper bound on n, so from now on assume that

$$\Delta_{n-1} > k. \tag{3.13}$$

We first show that

$$\delta' \le \delta\left(1 - \frac{2-\delta}{2-\delta^2}\left(\frac{2}{k} - \frac{32}{21k^2} - \frac{16}{7\delta k^3}\right)\right). \tag{3.14}$$

Let

$$y = 2\Delta - (k-r)(k-r+1), \quad \phi^* = \phi^*(k,r,\Delta) = \frac{2k}{2rk+y}.$$

By the definition of r_n,

$$k\delta(2k-k\delta-1)\le y\le k\delta(2k-k\delta+1).$$

Hence

$$\phi^*\ge\frac{2k}{2k(k-k\delta+1)+2\delta k^2-k\delta(k\delta-1)}=\frac{2}{(2-\delta^2)k+2+\delta}\ge\frac{1}{k+1},$$

so (3.11) holds. Iterating (3.12) gives

$$\theta_1\le 2^{1-j}\theta_j+\sum_{h=1}^{j-1}2^{1-h}(h^2-h)\frac{\phi^*}{4kr}\le 2^{1-j}\theta_j+\frac{2\phi^*}{kr}\le\frac{2^{1-j}}{r}+\frac{2\phi^*}{kr}.$$

Next, (3.13) implies $y\ge 2k-2$. Since $\sqrt{2k-2}\le k/3$, we always have $j\ge\sqrt{2k-2}$ (since j is maximal satisfying (3.8)) and so for $k\ge 1000$

$$\frac{2^{1-j}}{r}\le\frac{2^{1-\sqrt{2k-2}}}{r}\le\frac{0.071}{k^4r}.$$

Also, $\delta\le\frac12(1-1/k)$ implies

$$\phi^*\le\frac{2}{(2-\delta^2)k-\delta}\le\frac{8}{7k+1/k}<\frac{8}{7k}-\frac{0.16}{k^3},$$

and thus

$$\theta_1\le\frac{0.071}{k^4r}+\frac{16}{7k^2r}-\frac{0.32}{k^4r}\le\frac{16}{7k^2r}.$$

Since $\Delta\ge k$,

$$k^2+k-2\Delta=(k-\Delta/k)^2+k-(\Delta/k)^2\le(k-\delta k)(k-\delta k+1).$$

Therefore, from $k-\delta k\le r\le k-\delta k+1$ and the upper bound on θ_1,

$$\begin{aligned}\Delta'&=\Delta-k+\frac{\phi^*+\theta_1}{2}(2kr-y)\\&\le\Delta-k+\frac{\phi^*}{2}(2kr-y)+\frac{8}{7k^2}\left(r-1+\frac{k^2+k-2\Delta}{r}\right)\\&\le\Delta-2k+4k^2\frac{r}{2rk+y}+\frac{16(1-\delta)}{7k}.\end{aligned}\tag{3.16}$$

Next we establish

$$\frac{1-\delta}{2k}\le\frac{r}{2rk+y}\le\frac{1-\delta}{(2-\delta^2)k}+\frac{\delta}{(2-\delta^2)^2k^2}.\tag{3.17}$$

As a function of the real variable r, $\frac{r}{2rk+y}$ has positive second derivative and a minimum at $r = r_0 := \sqrt{k^2+k-2\Delta}$. Therefore, on the interval $[k-k\delta, k-k\delta+1]$, the maximum occurs at one of the endpoints. When $\delta > 1/\sqrt{k}$, $r_0 \le k-k\delta$, so the minimum occurs at $r = k-k\delta$. When $\frac{1}{k} \le \delta \le 1/\sqrt{k}$, $k-k\delta \le r_0 \le k-k\delta+1$, so the minimum occurs at $r = r_0$. At $r = k-k\delta+1$,

$$\frac{r}{2rk+y} = \frac{1-\delta+\frac{1}{k}}{(2-\delta^2)k+2+\delta} = \frac{1-\delta}{(2-\delta^2)k}\left(1+\frac{\delta}{k(1-\delta)(2-\delta^2)+2-\delta-\delta^2}\right),$$

so (3.17) holds for this r. When $r = k-k\delta$,

$$\frac{r}{2rk+y} = \frac{1-\delta}{(2-\delta^2)k-\delta} = \frac{1-\delta}{(2-\delta^2)k}\left(1+\frac{\delta}{(2-\delta^2)k-\delta}\right).$$

Since $(2-\delta^2)k-\delta > (2-\delta^2)k-\delta k(2-\delta^2) = (2-\delta^2)(1-\delta)k$, (3.17) holds for this r as well. Lastly, when $\frac{1}{k} \le \delta \le 1/\sqrt{k}$ and $r = r_0$,

$$\frac{r}{2rk+y} = \frac{1}{4k+1-2\sqrt{k^2+k-2\Delta}}.$$

Also,

$$\begin{aligned}\big(k+1/2-(\delta+\delta^2)k\big)^2 &= k^2+k+\tfrac{1}{4}-k(2k+1)(\delta+\delta^2)+k^2(\delta+\delta^2)^2\\ &\le k^2+k+\tfrac{1}{4}-2k^2(\delta+\delta^2)+k^2(\delta+\delta^2)^2\\ &= k^2+k-2\delta k^2+\tfrac{1}{4}-k^2(\delta^2-2\delta^3-\delta^4)\\ &< k^2+k-2\delta k^2.\end{aligned}$$

Therefore,

$$\frac{r}{2rk+y} \ge \frac{1}{4k+1-2(k+1/2-(\delta+\delta^2)k)} = \frac{1}{2k(1+\delta+\delta^2)} > \frac{1-\delta}{2k}.$$

This proves (3.17).

By (3.16) and (3.17), plus the inequality $\frac{(1-\delta)(2-\delta^2)}{2-\delta} \le 1$, we have

$$\begin{aligned}\delta' &\le \delta - \frac{2}{k} + \frac{4-4\delta}{(2-\delta^2)k} + \frac{4\delta}{(2-\delta^2)^2k^2} + \frac{16(1-\delta)}{7k^3}\\ &= \delta\left(1-\frac{4-2\delta}{(2-\delta^2)k}+\frac{4}{(2-\delta^2)^2k^2}\right)+\frac{16(1-\delta)}{7k^3}\\ &\le \delta\left(1-\frac{2-\delta}{2-\delta^2}\left(\frac{2}{k}-\frac{32}{21k^2}\right)\right)+\frac{16}{7k^3}\,\frac{2-\delta}{2-\delta^2}\\ &= \delta\left(1-\frac{2-\delta}{2-\delta^2}\left(\frac{2}{k}-\frac{32}{21k^2}-\frac{16}{7\delta k^3}\right)\right).\end{aligned}$$

This concludes the proof of (3.14). We now use (3.14) to bound Δ_n and C_n. Let $\beta = \frac{2}{k} - \frac{32}{21k^2}$, $c = \frac{16}{7k^3}$ and $\beta' = \beta - c/\delta$. The differential equation analogous to (3.14) is approximately $dy/dx = -\beta y \frac{2-y}{2-y^2}$, which has the implicit solution $y + \log y + \log(2-y) = -\beta x + C$ (this serves only as a motivation for the next inequality). Let

$$\delta'' = \delta\left(1 - \frac{2-\delta}{2-\delta^2}\beta'\right).$$

Since $y + \log y + \log(2-y)$ is increasing on $(0, 1/2]$, (3.14) gives

$$\begin{aligned}\delta' + \log\delta' + \log(2-\delta') &\le \delta'' + \log\delta'' + \log(2-\delta'') \\ &= \delta + \log\delta + \log(2-\delta) - \frac{2\delta-\delta^2}{2-\delta^2}\beta' + \log\left[\left(1 - \frac{2-\delta}{2-\delta^2}\beta'\right)\left(\frac{2-\delta''}{2-\delta}\right)\right].\end{aligned}$$

Write

$$T = -\frac{2\delta-\delta^2}{2-\delta^2}\beta' + \log\left(1 - \frac{2-\delta}{2-\delta^2}\beta'\right) + \log\left(\frac{2-\delta''}{2-\delta}\right).$$

Using

$$\frac{2-\delta''}{2-\delta} = 1 + \frac{\delta\beta'}{2-\delta^2} \qquad \text{and} \qquad \log(1+x) \le x - \frac{1}{2}x^2 + \frac{1}{3}x^3,$$

we obtain

$$\begin{aligned}T &\le -\beta' - \frac{(\beta')^2}{2(2-\delta^2)^2}\left((2-\delta)^2 + \delta^2\right) + \frac{(\beta')^3}{3(2-\delta^2)^3}\left(-(2-\delta)^3 + \delta^3\right) \\ &\le -\beta' - \frac{2}{5}(\beta')^2 \\ &\le -\beta - \frac{2}{5}\beta^2 + \frac{c(1+0.8\beta)}{\delta}.\end{aligned}$$

The minimum of $\frac{(2-\delta)^2+\delta^2}{2(2-\delta^2)^2}$ is actually $0.401\ldots$. Therefore

$$\delta' + \log\delta' + \log(2-\delta') \le \delta + \log\delta + \log(2-\delta) - \beta - 0.4\beta^2 + \frac{c(1+0.8\beta)}{\delta}.$$

Iteration of the above inequality yields

$$\begin{aligned}\delta_n + \log\delta_n + \log(2-\delta_n) \le \delta_1 + \log\delta_1 + \log(2-\delta_1) - (n-1)(\beta + 0.4\beta^2) \\ + c(1+1.6/k)\left(\frac{1}{\delta_1} + \cdots + \frac{1}{\delta_{n-1}}\right).\end{aligned}$$

By (3.13) and (3.14),

$$\delta_{i+1} \le \delta_i(1-\alpha), \quad \alpha = \frac{6}{7}(\beta - kc). \tag{3.18}$$

By (3.13) again, this gives

$$c(1+1.6/k)\left(\frac{1}{\delta_1}+\cdots+\frac{1}{\delta_{n-1}}\right)\le\frac{c(1+1.6/k)}{\alpha\delta_{n-1}}\le\frac{1.34}{k}.$$

Therefore,

$$\delta_n\le\frac{\delta_1(2-\delta_1)e^{\delta_1}}{(2-\delta_n)e^{\delta_n}}e^{-(n-1)(\beta+0.4\beta^2)+1.34/k}. \tag{3.19}$$

Next,

$$\beta+0.4\beta^2\ge\frac{2}{k}-\frac{32}{21k^2}+\frac{0.4}{k^2}\left(2-\frac{32/21}{1000}\right)^2\ge\frac{2}{k}.$$

From (3.13) and the inequality $1+x\le e^x$, we have

$$\begin{aligned}\delta_1(2-\delta_1)e^{\delta_1}&=\frac{e^{1/2}}{2}\left(1-\tfrac{1}{k}\right)\left(\tfrac{3}{2}+\tfrac{1}{2k}\right)e^{-1/(2k)}\le\frac{3}{4}e^{1/2-7/(6k)},\\ \frac{e^{-\delta_n}}{2-\delta_n}&\le\tfrac{1}{2}e^{\delta_n/(2-\delta_n)-\delta_n}\le\frac{1}{2}e^{\frac{0.49}{k}}.\end{aligned}$$

Putting these together with (3.19) gives

$$\delta_n\le\frac{3}{8}e^{1/2-2n/k+1.69/k}.$$

To bound the constants C_n, take $\omega=0.06>1/(3\log k)$, so that

$$V^{k+1}=(300k^3\log k)^{k+1}\le k^{4.11k}=:W.$$

We next prove that

$$W^{\Delta_{n-1}-\Delta_n}>k^{3k}1.06^{4k(n-1)+k^2}\qquad(n\le1.97k+1). \tag{3.20}$$

By (3.14),

$$\delta_{n-1}-\delta_n\ge\frac{2\delta_{n-1}}{k}\left(\frac{2-\delta_{n-1}}{2-\delta_{n-1}^2}-0.002\right). \tag{3.21}$$

By the top line of (3.16) and (3.17),

$$\begin{aligned}\delta_m&\ge\delta_{m-1}-\frac{2}{k}+\frac{4r}{2kr+y}\\ &\ge\delta_{m-1}-\frac{2}{k}+4\,\frac{1-\delta_{m-1}}{2k}=\delta_{m-1}\left(1-\frac{2}{k}\right),\end{aligned}$$

which implies

$$\delta_{n-1} \ge (1-2/k)^{n-2}\delta_1 \ge \tfrac12(1-2/k)^{n-1} \ge \tfrac12 e^{-\frac{2}{k-2}(n-1)} \ge 0.0096476 := \bar\delta.$$

The right side of (3.21) is increasing in δ_{n-1}, so

$$\delta_{n-1}-\delta_n \ge \frac{2\bar\delta}{k}\left(\frac{2-\bar\delta}{2-\bar\delta^2} - 0.002\right) \ge \frac{0.01916}{k}.$$

Therefore, $W^{\Delta_{n-1}-\Delta_n} \ge k^{0.0787k^2}$. On the other hand,

$$k^{3k}1.06^{4k(n-1)+k^2} \le k^{k^2(0.003+8.88\log(1.06)/\log(1000))} \le k^{0.078k^2}.$$

This proves (3.20). Let $n_0 = \lfloor 1.97k \rfloor + 1$. By (3.20) and Lemma 3.5,

$$C_{n_0} \le W^{\Delta_1-\Delta_{n_0}}k! \le W^{\frac12 k^2-\Delta_{n_0}}$$

and for $n > n_0$

$$C_n \le k^{3k}1.06^{4k(n-1)+k^2}W^{\Delta_{n-1}-\Delta_n}C_{n-1}.$$

Iterating this last inequality gives, for $n > n_0$,

$$\begin{aligned} C_n &\le W^{\frac12 k^2}k^{3k(n-n_0)}1.06^{(n-n_0)k^2+4k(n_0+\cdots+n-1)} \\ &\le W^{\frac12 k^2}k^{3k(n-1.97k)}1.06^{(n-1.97k)k^2+2k(n^2-n-(1.97k)^2+1.97k)} \\ &\le k^{2.055k^3-5.91k^2+3nk}1.06^{nk^2+2(n^2-n)k-9.7278k^3}. \end{aligned}$$

This finishes the proof of Lemma 3.6. □

Proof of Theorem 3. Suppose first that $k \ge 1000$. Every permissible s can be written as $s = nk+u$ where $0 \le u \le k$ and $n \le \frac{k}{2}(\frac12 + \log\frac{3k}{8})$. By Lemma 3.6 and Hölder's inequality,

$$J_{s,k}(P) \le k^{2.055k^3-5.91k^2+3s}1.06^{sk+2s^2/k-9.7278k^3}P^{2s-\frac12 k(k+1)+\Delta},$$

where

$$\Delta = \frac38 k^2 e^{1/2-2n/k+1.69/k}\left[1-\tfrac{u}{k}+\tfrac{u}{k}e^{-2/k}\right].$$

Lastly,

$$1-\tfrac{u}{k}+\tfrac{u}{k}e^{-2/k} \le 1-\tfrac{2u}{k^2}+\tfrac{2u}{k^3} \le e^{-2u/k^2+2u/k^3},$$

thus $\Delta \le \frac38 k^2 e^{1/2-2s/k^2+1.7/k}$.

Next, suppose $129 \le k \le 1001$. Start with $\Delta_1 = \frac12 k^2(1-1/k)$, successively choose r_n near $\sqrt{k^2+k-2\Delta_n}$ satisfying (3.11), and set $\Delta_{n+1} = \delta_0(k, r_n, \Delta_n)$. Also take C_n as in Lemma 3.5, where we define ω by

$$\frac{1}{3\log k} \le \omega \le \frac{1}{2\log k + (4/3)\log\log k}, \quad e^{1.5+1.5/\omega} = \frac{18}{\omega}k^3\log k$$

and take $\eta = 1+\omega$. To see that ω is well-defined, let $h(\omega) = e^{1.5+1.5/\omega} - \frac{18}{\omega}k^3 \log k$, $\omega_0 = \frac{1}{3\log k}$ and $\omega_1 = 1/(2\log k + (4/3)\log\log k)$. It is easy to verify that $h(\omega_0) > 0$, $h(\omega_1) < 0$ and $h'(\omega) < 0$ for $\omega \in [\omega_0, \omega_1]$.

If $\Delta_{n+1} \le \frac{k^2}{1000} \le \Delta_n$, take

$$s = \left\lceil \left(n + \frac{\Delta_n - k^2/1000}{\Delta_n - \Delta_{n+1}} \right) k \right\rceil .$$

By Hölder's inequality,

$$J_{s,k}(P) \le C_{n+1} P^{2s - \frac{1}{2}k(k+1) + 0.001k^2}.$$

A straightforward computer computation verifies the claimed bounds on s and C_n. The program is listed in the Appendix. □

Remarks. One can obtain slightly better values for Δ_n using a variant of the iterative scheme embodied in Lemmas 3.2 and 3.3. For example, this alternate method would produce bounds valid with $\rho = 3.20354$ for $129 \le k \le 199$. The improvement, however, becomes negligible for large k. Instead of working with $K_s(P,Q;\mathbf{\Psi};q)$, we work on bounding $K_{s,d}(P,Q;\mathbf{\Psi};q)$, the number of solutions of

$$\sum_{i=1}^{k-d} (\Psi_j(z_i) - \Psi_j(w_i)) + q^j \sum_{i=1}^{s} (x_i^j - y_i^j) = 0 \qquad (1 \le j \le k),$$
$$1 \le z_i, w_i \le P; \quad 1 \le x_i, y_i \le Q.$$

Define $L_{s,d}(P,Q;\mathbf{\Psi};p,q,r)$ similarly. In Lemma 3.2, the variables $z_{k-d+1}, \dots, z_k$ and $w_{k-d+1}, \dots, w_k$ are not utilized in the argument because $\Psi_j(z) = 0$ for $j \le d$. Following the proof of Lemma 3.2 with the new quantities gives

Lemma 3.2'. *With the same hypotheses as Lemma 3.2,*

$$K_{s,d}(P,Q;\mathbf{\Psi};q) \le 4k^3 k! p^{2s+\frac{1}{2}(r-d)(r-d+1)} L_{s,d}(P,Q;\mathbf{\Phi};p,q,r).$$

Likewise, following the proof of Lemma 3.3 and using Hölder's inequality at the end gives

Lemma 3.3'. *Under the hypotheses of Lemma 3.3,*

$$L_{s,d}(P;Q;\mathbf{\Phi};p,q,r) \le (2P)^{k-d} \max\Big[k^{k-d} J_{s,k}(Q),$$
$$2p^{-r(k-d)} J_{s,k}(Q)^{\frac{k-d-2}{2(k-d-1)}} K_{s,d+1}(P,Q;\mathbf{\Upsilon};pq)^{\frac{k-d}{2(k-d-1)}}\Big].$$

In Lemma 3.4, the definition of ϕ_J changes to

$$\phi_J = \frac{1}{2r} + \frac{k^2+k+r^2-r-2\Delta-2rJ}{4r(k-J)}\phi_{J+1} \quad (1 \le J \le j-1),$$

and this produces slightly smaller values for ϕ_1. The only downside is that the analysis of the numbers δ_n (see Lemma 3.6) becomes more complicated.

4. Incomplete systems and smooth Weyl sums.

The object of this section is to obtain explicit upper bounds on $J_{s,k,h}(\mathscr{B})$, the number of solutions of

$$\sum_{i=1}^{s}(x_i^j - y_i^j) = 0 \qquad (h \le j \le k); x_i, y_i \in \mathscr{B}, \tag{4.1}$$

where $\mathscr{B} = \mathscr{C}(P,R) = \{1 \le n \le P : p|n \implies \sqrt{R} < p \le R\}$. Suppose $k \ge h \ge 2$ and set $t = k-h+1$. For a t-tuple $\boldsymbol{x} = (x_1, \cdots, x_t)$, let

$$J(\boldsymbol{x}) = \det\big(jx_i^{j-1}\big)_{\substack{1\le i\le t\\ h\le j\le k}} = \frac{k!}{(h-1)!}(x_1\cdots x_t)^{h-1}\prod_{1\le i<j\le t}(x_i - x_j) \tag{4.2}$$

be the Jacobian of the functions $\sum_{i=1}^t x_i^j$ $(h \le j \le k)$. The notation $x\mathscr{D}(Q)y$ means that there is some $d|x$ with $d \le Q$ and $s_0(x/d)|s_0(y)$. For $\boldsymbol{\alpha} = (\alpha_h, \cdots, \alpha_k)$, define the exponential sum

$$f(\boldsymbol{\alpha}) = f(\boldsymbol{\alpha}; P, R) = \sum_{x\in\mathscr{C}(P,R)} e(\alpha_h x^h + \cdots + \alpha_k x^k)$$

so that

$$J_{s,k,h}(\mathscr{C}(P,R)) = \int_{\mathbb{U}^t} |f(\boldsymbol{\alpha})|^{2s}\, d\boldsymbol{\alpha}.$$

Our main lemma is very similar to the the "fundamental lemma" (Lemma 3.1 of [34]). However, we do not perform "repeat efficient differencing" as in [29], [30], [34], and Lemma 3.4 of this paper.

Lemma 4.1. *Suppose*

$$\begin{aligned} &k \ge h \ge 8, \quad t = k-h+1, \quad s \ge t+1, \quad h \le r \le k; \\ &P > (8s)^{20}, \qquad R = P^\eta > k^2, \qquad |\mathscr{C}(P,R)| \ge P^{1/2}. \end{aligned} \tag{4.3}$$

Then

$$J_{s,k,h}(\mathscr{C}(P,R)) \le \max\bigg[\Big((8s)^2(22t^2)^{\frac{2}{\eta}}P^{1/r}\Big)^{s-t} k^t|\mathscr{C}(P,R)|^s, 4k^{2t(\frac{1}{r\eta}+1)}$$
$$\times |\mathscr{C}(P,R)|^t(P^{\frac1r}R)^{\frac12(r-h)(r-h+1)}\bigg\{\sum_{P^{\frac1r}<q\le P^{\frac1r}R} J_{s-t,k,h}(\mathscr{C}(P/q,R))^{\frac{1}{2s-2t}}\bigg\}^{2s-2t}\bigg].$$

Proof. For short, let $S_0 = J_{s,k,h}(\mathscr{C}(P,R))$, $\boldsymbol{x} = (x_1, \dots, x_t)$, $\boldsymbol{y} = (y_1, \dots, y_t)$ and $\boldsymbol{\alpha} = (\alpha_h, \dots, \alpha_k)$. We divide the solutions of (4.1) into four classes: S_1 counts the solutions with $\min(x_i, y_i) \le P^{1/5}$ for some i; S_2 counts the solutions with $x_i = x_j$ or $y_i = y_j$ for some $1 \le i < j \le t$; S_3 counts solutions not counted by S_1 or S_2, and with $x_i\mathscr{D}(P^{1/r})J(\boldsymbol{x})$ or $y_i\mathscr{D}(P^{1/r})J(\boldsymbol{y})$ for some $i > t$; S_4 (which will be the main term) counts the solutions not counted by S_1, S_2 or S_3.

Evidently $S_0 \le 4\max(S_1, S_2, S_3, S_4)$. If S_1 is the largest, then by a trivial estimate and Hölder's inequality,

$$\begin{aligned} S_0 \le 4S_1 &\le 8s \int_{\mathbb{U}^t} |f(\boldsymbol{\alpha})^{2s-1} f(\boldsymbol{\alpha}; P^{1/5}, R)| \, d\boldsymbol{\alpha} \\ &\le 8sP^{1/5} \left(\int_{\mathbb{U}^t} |f(\boldsymbol{\alpha})^{2s}| \, d\boldsymbol{\alpha} \right)^{1-\frac{1}{2s}} \\ &= 8sS_0^{1-1/2s} P^{1/5}. \end{aligned}$$

Therefore, $S_0 \le (8sP^{1/5})^{2s}$. However, counting only the trivial solutions of (4.1) (those with $x_i = y_i$ for every i) and using (4.3) gives

$$S_0 \ge |\mathscr{C}(P, R)|^s \ge P^{s/2} > (8sP^{1/5})^{2s}, \tag{4.4}$$

giving a contradiction.

If S_2 is the largest, then by Hölder's inequality,

$$\begin{aligned} S_0 \le 4S_2 &\le 8\binom{t}{2} \int_{\mathbb{U}^t} |f(\boldsymbol{\alpha})^{2s-2} f(2\boldsymbol{\alpha})| \, d\boldsymbol{\alpha} \\ &\le 4t^2 \left(\int_{\mathbb{U}^t} |f(\boldsymbol{\alpha})|^{2s} d\boldsymbol{\alpha} \right)^{1-\frac{1}{s}} \left(\int_{\mathbb{U}^t} |f(2\boldsymbol{\alpha})|^{2s} \, d\boldsymbol{\alpha} \right)^{\frac{1}{2s}} \\ &= 4t^2 S_0^{1-\frac{1}{2s}}. \end{aligned}$$

By (4.3), $S_0 \le (4t^2)^{2s} < (8s)^{4s} < P^{s/2}$, contradicting (4.4). It follows that $S_0 \le 4\max(S_3, S_4)$.

Suppose next that $S_3 = \max(S_3, S_4)$. From (4.2), we have $J(\boldsymbol{x}) \ne 0$ and $J(\boldsymbol{y}) \ne 0$ for each solution $(x_1, y_1, \dots, x_s, y_s)$ of (4.1) counted in S_3. Let

$$\mathscr{S}(\boldsymbol{x}) = \{w \in \mathscr{C}(P, R) : w\mathscr{D}(P^{1/r})J(\boldsymbol{x})\}$$

and define

$$H(\boldsymbol{\alpha}) = \sum_{\substack{\boldsymbol{x}: J(\boldsymbol{x}) \ne 0 \\ x_i \in \mathscr{C}(P,R)}} \sum_{w \in \mathscr{S}(\boldsymbol{x})} e\left(\sum_{j=h}^{k} \alpha_j (w^j + x_1^j + \cdots + x_t^j) \right).$$

By the Cauchy-Schwarz inequality ,

$$\begin{aligned} S_0 \le 4S_3 &\le 8(s-t) \int_{\mathbb{U}^t} |H(\boldsymbol{\alpha}) f(\boldsymbol{\alpha})^{2s-t-1}| \, d\boldsymbol{\alpha} \\ &\le 8s \left(\int_{\mathbb{U}^t} |f(\boldsymbol{\alpha})|^{2s} \, d\boldsymbol{\alpha} \right)^{1/2} \left(\int_{\mathbb{U}^t} |H^2(\boldsymbol{\alpha}) f(\boldsymbol{\alpha})^{2s-2t-2}| \, d\boldsymbol{\alpha} \right)^{1/2} \\ &= 8sS_0^{1/2} \left(\int_{\mathbb{U}^t} |H^2(\boldsymbol{\alpha}) f(\boldsymbol{\alpha})^{2s-2t-2}| \, d\boldsymbol{\alpha} \right)^{1/2}. \end{aligned}$$

Therefore,

$$S_0 \le (8s)^2 \int_{\mathbb{U}^t} |H(\boldsymbol{\alpha})^2 f(\boldsymbol{\alpha})^{2s-2t-2}|\, d\boldsymbol{\alpha},$$

and the integral on the right is the number of solutions of

$$\begin{gathered}\sum_{i=1}^{s-1}(x_i^j - y_i^j) + (dw)^j - (ez)^j = 0 \qquad (h \le j \le k)\\ x_i, y_i \in \mathscr{C}(P,R);\ d,e \in \mathscr{C}(P^{1/r},R);\ J(\boldsymbol{x}) \ne 0, J(\boldsymbol{y}) \ne 0;\\ w \in \mathscr{C}(P/d,R), z \in \mathscr{C}(P/e,R);\ s_0(w)|J(\boldsymbol{x}), s_0(z)|J(\boldsymbol{y}).\end{gathered}$$

Writing

$$G_g(\boldsymbol{\alpha}) = \sum_{\substack{\boldsymbol{x}: J(\boldsymbol{x})\ne 0\\ g|J(\boldsymbol{x})}} e\left(\sum_{j=h}^k \alpha_j(x_1^j + \cdots + x_t^j)\right),$$

$$\mathscr{G}(\boldsymbol{\alpha}) = \sum_{\substack{g\in\mathscr{C}(P,R)\\ \mu^2(g)=1}} G_g(\boldsymbol{\alpha}) \sum_{d\in\mathscr{C}(P^{1/r},R)} \sum_{\substack{w\in\mathscr{C}(P/d,R)\\ s_0(w)=g}} e\left(\alpha_h(dw)^h + \cdots + \alpha_k(dw)^k\right),$$

it follows that

$$S_0 \le (8s)^2 \int_{\mathbb{U}^t} |\mathscr{G}(\boldsymbol{\alpha})^2 f(\boldsymbol{\alpha})^{2s-2t-2}|\, d\boldsymbol{\alpha}. \tag{4.5}$$

By the Cauchy-Schwarz inequality ,

$$|\mathscr{G}(\boldsymbol{\alpha})|^2 \le \left(\sum_g |G_g(\boldsymbol{\alpha})|^2\right)\left(\sum_g \left|\sum_{d,w} 1\right|^2\right).$$

Next,

$$\begin{aligned}\sum_g \left|\sum_{d,w} 1\right|^2 &\le \sum_g \left(P^{1/r}|\{w \le P : s_0(w) = g\}|\right) \sum_{\substack{w\in\mathscr{C}(P,R)\\ g|w}} \sum_{d\in\mathscr{C}(P/w,R)} 1\\ &\le P^{1/r}\sum_g |\{w \le P : s_0(w) = g\}| \sum_{\substack{n\in\mathscr{C}(P,R)\\ g|n}} d_2(n)\\ &\le P^{1/r} \max_{\substack{g\in\mathscr{C}(P,R)\\ \mu^2(g)=1}} |\{w \le P : s_0(w) = g\}| \sum_{n\in\mathscr{C}(P,R)} d_2^2(n).\end{aligned}$$

For any $m \in \mathscr{C}(P,R)$, $\tau(m) \le 2^{\Omega(m)} \le 2^{2/\eta}$. Any $g \in \mathscr{C}(P,R)$ with $\mu^2(g) = 1$ can be written as $g = p_1 \cdots p_n$, where $p_1, \ldots, p_n$ are distinct primes each larger than

$\sqrt{R}$, and $0 \le n \le 2/\eta$. Then

$$\begin{aligned}|\{w \le P : s_0(w) = g\}| &= |\{\boldsymbol{u} : u_1 \log p_1 + \cdots + u_n \log p_n \le \log P : u_i \ge 1\ \forall i\}| \\ &\le |\{\boldsymbol{u} : u_1 + \cdots + u_n \le 2/\eta : u_i \ge 1\ \forall i\}| \\ &= \binom{\lfloor 2/\eta \rfloor}{n} < 2^{2/\eta}.\end{aligned}$$

Therefore,

$$|\mathscr{G}(\boldsymbol{\alpha})|^2 \le 2^{6/\eta} P^{1/r} |\mathscr{C}(P,R)| \sum_{\substack{g \in \mathscr{C}(P,R) \\ \mu^2(g)=1}} |G_g(\boldsymbol{\alpha})|^2,$$

whence by (4.5),

$$S_0 \le (8s)^2 2^{6/\eta} P^{1/r} |\mathscr{C}(P,R)| V, \tag{4.6}$$

where

$$V = \int_{\mathbb{U}^t} \sum_{\substack{g \in \mathscr{C}(P,R) \\ \mu^2(g)=1}} \left| G_g(\boldsymbol{\alpha})^2 f(\boldsymbol{\alpha})^{2s-2t-2} \right| d\boldsymbol{\alpha}.$$

Here V counts the solutions $(x_1, y_1, \dots, x_{s-1}, y_{s-1}, g)$ of

$$\sum_{i=1}^{s-1} (x_i^j - y_i^j) = 0 \qquad (h \le j \le k)$$

$$x_i, y_i, g \in \mathscr{C}(P,R);\ J(\boldsymbol{x}) \ne 0, J(\boldsymbol{y}) \ne 0; \quad \mu^2(g) = 1, g|J(\boldsymbol{x}), g|J(\boldsymbol{y}).$$

Clearly

$$V \le J_{s-1,k,h}(\mathscr{C}(P,R)) \max_{J(\boldsymbol{x}) \ne 0} \left| \{g \in \mathscr{C}(P,R), \mu^2(g) = 1, g|J(\boldsymbol{x})\} \right|.$$

Using (4.2), $\sqrt{R} > k$ and $\mu^2(g) = 1$, $g|J(\boldsymbol{x})$ implies $g|J^*(\boldsymbol{x})$, where

$$J^*(\boldsymbol{x}) = x_1 \cdots x_t \prod_{1 \le i < j \le t} (x_i - x_j).$$

Since $|J^*(\boldsymbol{x})| < P^{t(t+1)/2}$, $J^*(\boldsymbol{x})$ has at most $t(t+1)/\eta$ distinct prime factors $> \sqrt{R}$. If $g|J^*(\boldsymbol{x})$, then g is a product of n of these primes, where $0 \le n \le 2/\eta$. The number of such g is at most

$$\sum_{0 \le n \le 2/\eta} \binom{\lfloor (t^2+t)/\eta \rfloor}{n} \le \sum_{0 \le n \le 2/\eta} \frac{(2t^2/\eta)^n}{n!} \le t^{4/\eta} \sum_{n=0}^{\infty} \frac{(2/\eta)^n}{n!} = (et^2)^{2/\eta}.$$

From (4.6) we conclude that

$$S_0 \le (8s)^2 (8et^2)^{2/\eta} |\mathscr{C}(P,R)| P^{1/r} J_{s-1,k,h}(\mathscr{C}(P,R)).$$

Lastly, applying Hölder's inequality, we have

$$\begin{aligned} J_{s-1,k,h}(\mathscr{C}(P,R)) &\le J_{s,k,h}(\mathscr{C}(P,R))^{1-\frac{1}{s-t}} J_{t,k,h}(\mathscr{C}(P,R))^{\frac{1}{s-t}} \\ &= S_0^{1-\frac{1}{s-t}} J_{t,k,h}(\mathscr{C}(P,R))^{\frac{1}{s-t}}. \end{aligned}$$

We have $J_{t,k,h}(\mathscr{C}(P,R)) \le k^t|\mathscr{C}(P,R)|^t$, which follows for instance from Lemma 2.4 (let p be a prime $> tP$, fix $y_1, \dots, y_t$ and for each $\boldsymbol{u}$ the number of $\boldsymbol{x}$ with $\sum x_i^j \equiv u_j \pmod p (h \le j \le k)$ is $\le k^t$). This proves the lemma in the case $S_3 \ge S_4$.

For the last case, suppose $S_4 = \max(S_1, S_2, S_3, S_4)$. For every solution of (4.1) counted by S_4, each $x_i > P^{1/r}$ and $y_i > P^{1/r}$ and neither $x_i \mathscr{D}(P^{1/r})J(\boldsymbol{x})$ nor $y_i \mathscr{D}(P^{1/r})J(\boldsymbol{y})$ for $i > t$. Fix $i > t$ and let q be the greatest divisor of x_i with the property that $(q, J(\boldsymbol{x})) = 1$. If $q \le P^{1/r}$, then $x_i \mathscr{D}(P^{1/r})J(\boldsymbol{x})$, a contradiction. Hence $q > P^{1/r}$, and since every prime divisor of q is $\le R$, there is a divisor q_i of x_i with $q_i > P^{1/r}$, $q_i \in \mathscr{C}(P^{1/r}R, R)$ and $(q_i, J(\boldsymbol{x})) = 1$. Likewise, each y_i has a divisor p_i with $p_i > P^{1/r}$, $p_i \in \mathscr{C}(P^{1/r}R, R)$ and $(p_i, J(\boldsymbol{y})) = 1$. Therefore $S_0 \le 4T$, where T is the number of solutions of

$$\begin{gathered} \sum_{i=1}^{t}(x_i^j - y_i^j) + \sum_{i=1}^{s-t}((q_i u_i)^j - (p_i v_i)^j) = 0 \qquad (h \le j \le k) \\ x_i, y_i \in \mathscr{C}(P,R);\; u_i \in \mathscr{C}(P/q_i, R), v_i \in \mathscr{C}(P/p_i, R); \\ p_i, q_i \in \mathscr{C}(P^{1/r}R, R); p_i, q_i > P^{1/r}; (q_i, J(\boldsymbol{x})) = (p_i, J(\boldsymbol{y})) = 1. \end{gathered}$$

Let

$$F_q(\boldsymbol{\alpha}) = \sum_{\boldsymbol{x}:(q,J(\boldsymbol{x}))=1} e\left(\sum_{j=h}^{k} \alpha_j (x_1^j + \cdots + x_t^j)\right).$$

Given $q_1, p_1, \dots q_{s-t}, p_{s-t}$, let

$$\widetilde{p} = p_1 \cdots p_{s-t}, \qquad \widetilde{q} = q_1 \cdots q_{s-t}$$

and set

$$\begin{aligned} X_i(\boldsymbol{\alpha}) &= \left|F_{\widetilde{q}}(\boldsymbol{\alpha})^2 f((q_i^h \alpha_h, \cdots, q_i^k \alpha_k); P/q_i, R)^{2s-2t}\right|, \\ Y_i(\boldsymbol{\alpha}) &= \left|F_{\widetilde{p}}(\boldsymbol{\alpha})^2 f((p_i^h \alpha_h, \cdots, p_i^k \alpha_k); P/p_i, R)^{2s-2t}\right|. \end{aligned}$$

Then, by Hölder's inequality, we have

$$\begin{aligned} S_0 &\le 4 \sum_{\boldsymbol{p},\boldsymbol{q}} \int_{\mathbb{U}^t} \prod_{i=1}^{s-t} (X_i(\boldsymbol{\alpha}) Y_i(\boldsymbol{\alpha}))^{\frac{1}{2s-2t}} \, d\boldsymbol{\alpha} \\ &\le 4 \sum_{\boldsymbol{p},\boldsymbol{q}} \prod_{i=1}^{s-t} \left(\int_{\mathbb{U}^t} X_i(\boldsymbol{\alpha})\, d\boldsymbol{\alpha}\right)^{\frac{1}{2s-2t}} \left(\int_{\mathbb{U}^t} Y_i(\boldsymbol{\alpha})\, d\boldsymbol{\alpha}\right)^{\frac{1}{2s-2t}}. \end{aligned}$$

We have $\int_{\mathbb{U}^t} X_i(\boldsymbol{\alpha})\,d\boldsymbol{\alpha} \le W(q_i)$ and $\int_{\mathbb{U}^t} Y_i(\boldsymbol{\alpha})\,d\boldsymbol{\alpha} \le W(p_i)$, where $W(q)$ is the number of solutions of

$$\begin{aligned} &\sum_{i=1}^{t}(x_i^j - y_i^j) + q^j \sum_{i=1}^{s-t}(u_i^j - v_i^j) = 0 \qquad (h \le j \le k) \\ &x_i, y_i \in \mathscr{C}(P,R); \quad u_i, v_i \in \mathscr{C}(P/q,R); \quad (q, J(\boldsymbol{x})J(\boldsymbol{y})) = 1. \end{aligned} \tag{4.7}$$

Thus

$$S_0 \le 4 \sum_{\boldsymbol{p},\boldsymbol{q}} \prod_{i=1}^{s-t} (W(q_i)W(p_i))^{\frac{1}{2s-2t}} = 4\Bigg(\sum_{\substack{q\in\mathscr{C}(P^{1/r}R,R) \\ q>P^{1/r}}} W(q)^{\frac{1}{2s-2t}} \Bigg)^{2s-2t}. \tag{4.8}$$

Next, by Proposition ZRD, for each possible $2t$-tuple $\boldsymbol{x}, \boldsymbol{y}$ in (4.7), the number of $\boldsymbol{u}, \boldsymbol{v}$ is at most $J_{s-t,k,h}(\mathscr{C}(P/q,R))$. By fixing $\boldsymbol{y}$, the number of possible $\boldsymbol{x}, \boldsymbol{y}$ is $\le |\mathscr{C}(P,R)|^t \max_{\boldsymbol{m}} \mathscr{B}(\boldsymbol{m})$, where $\mathscr{B}(\boldsymbol{m})$ is the number of solutions of the simultaneous congruences

$$\sum_{i=1}^{t} x_i^j \equiv m_j \pmod{q^j} \qquad (h \le j \le k)$$

with $1 \le x_i \le P$ and $(q, J(\boldsymbol{x})) = 1$. For each j, the number of possibilities for m_j modulo q^r is $\max(1, q^{r-j})$. Thus

$$\mathscr{B}(\boldsymbol{m}) \le q^{(r-h)(r-h+1)/2} \max_{\boldsymbol{n}} \mathscr{B}'(\boldsymbol{n}; q^r),$$

where $\mathscr{B}'(\boldsymbol{n}; q^r)$ is the number of solutions of

$$\sum_{i=1}^{t} x_i^j \equiv n_j \pmod{q^r} \qquad (h \le j \le k)$$

with $1 \le x_i \le q^r$ (recall $q^r \ge P$) and $(q, J(\boldsymbol{x})) = 1$. By the Chinese Remainder Theorem,

$$\mathscr{B}'(\boldsymbol{n}; q^r) \le \prod_{p^\ell \| q, p \text{ prime}} \mathscr{B}'(\boldsymbol{n}; p^{r\ell}),$$

and Lemma 2.4 gives $\mathscr{B}'(\boldsymbol{n}; p^{r\ell}) \le k!/(h-1)! \le k^t$. Since $\omega(q) \le 2/(r\eta) + 2$, we have $\mathscr{B}'(\boldsymbol{n}; q^r) \le k^{2t(1+1/(r\eta))}$. This gives

$$W(q) \le k^{2t(1+1/(r\eta))} q^{(r-h)(r-h+1)/2} |\mathscr{C}(P,R)|^t J_{s-t,k,h}(\mathscr{C}(P/q,R)).$$

Together with (4.8), this proves the lemma in the fourth case. $\square$

The optimal choice for r in the above lemma is close to h for the range of s that we are interested in. The next lemma gives some bounds achievable with Lemma 4.1.

Lemma 4.2. *Suppose that k,h and L are integers satisfying*

$$k \ge 60, \quad h \le k, \quad t = k-h+1 \le \tfrac{k}{6}, \quad 1 \le L \le h/2. \tag{4.9}$$

Let $\alpha = 1-1/h$. Suppose P, R and η are real numbers with

$$0 < \eta \le \frac{2}{3h}, \qquad R = P^{\eta} \ge \left(\frac{2}{\eta}\right)^3, \tag{4.10}$$

and

$$|\mathscr{C}(Q,R)| \ge Q^{1/2} \qquad (P^{1/3} \le Q \le P). \tag{4.11}$$

Then

$$J_{Lt,k,h}(\mathscr{C}(P,R)) \le (10\eta)^{tL((1/\eta+h)\alpha^{L-1}-h)} C_L (e^2R)^{\frac{t}{2}L(L-1)} P^{2Lt-\frac{t}{2}(h+k)+\Delta_L},$$

where

$$\begin{aligned}
\Delta_j &= \frac{t(t-1)}{2} + ht(1-1/h)^j \qquad (j \ge 1),\\
C_1 &= k^t, \quad C_\ell = \max_{2\le j\le \ell} e^{tE_j} \quad (\ell \ge 2),\\
E_j &= \alpha^{L-j}\left[\frac{4\log k}{\eta}(j-1) - \left(j - \frac{j-1}{h} - h + h\alpha^j\right)\log P\right].
\end{aligned}$$

Proof. For $1 \le j \le L$, define $P_j = P^{\alpha^{L-j}}$, $M_j = P^{\alpha^{L-j}} R^{-h(1-\alpha^{L-j})}$, $\eta_j = \frac{\log R}{\log M_j}$ and $\eta_j' = \frac{\log R}{\log P_j} = \alpha^{j-L}\eta$. By (4.9) and (4.10),

$$M_j \ge P^{\alpha^L} R^{-h(1-\alpha^L)} \ge P^{0.6} R^{-0.4h} \ge P^{\frac{1}{3}} \ge (2/\eta)^{1/\eta} \ge (3h)^{75} > (8Lt)^{20}. \tag{4.12}$$

Consequently, $\eta \le \eta_j' < \eta_j \le \eta_1 \le 3\eta$ for every j. For $M \ge 1$ let $H_j(M) = J_{tj,k,h}(\mathscr{C}(M,R))$. We prove by induction on j that

$$H_j(M) \le (10\eta)^{tj/\eta_1} C_j (e^2R)^{\frac{t}{2}j(j-1)} M^{2jt-\frac{t}{2}(h+k)+\Delta_j} \quad (M_j \le M \le P_j). \tag{4.13}$$

By (4.10), $R \ge (3h)^3 > k^3 > 90000$. By (4.11) and (4.12), when $M_1 \le M \le P$ we have $|\mathscr{C}(M,R)| \ge M^{1/2}$, so all of the hypotheses (4.3) of Lemma 4.1 hold (with M in place of P). Also, if $M_1 \le R^u \le P$ then $R \ge (2/\eta)^3 \ge (2u)^3$, hence the hypotheses of Lemma 2.3 hold. For $M \ge M_1$, as in the proof of Lemma 4.1 we have $H_1(M) \le k^t|\mathscr{C}(M,R)|^t$. Writing $\nu = \frac{\log M}{\log R}$, by Lemma 2.3

$$|\mathscr{C}(M,R)| \le M(2\nu)^{1/\nu} \le M(6\eta)^{1/\eta_1}, \tag{4.14}$$

so (4.13) holds for $j=1$. Next assume $j \ge 2$, (4.13) holds with j replaced by $j-1$, and assume $M_j \le M \le P_j$. We will apply Lemma 4.1 with $r=h$ and $P=M$. By the definition of M_j and P_j,

$$M_{j-1} \le M/q \le P_{j-1} \qquad \left(P^{1/h} < q \le P^{1/h}R\right).$$

By (4.9) and (4.10),

$$k(8jt)^2(22t^2)^{2/\nu} \le \left(k^{3/h}22t^2\right)^{2/\nu} < (27t^2)^{2/\nu} < k^{4/\nu} \le k^{4/\eta_j'}.$$

By Lemma 2.3 and $\nu \le 3\eta \le 2/h$,

$$4k^{2t(\frac{1}{h\nu}+1)}|\mathscr{C}(M,R)|^t \le 4M^t e^{\frac{t}{\nu}(\log(2\nu)+(2/h+2\nu)\log k)} \le 4M^t e^{\frac{t}{\nu}\log(3.13\nu)}.$$

Since $e^{\frac{t}{\nu}\log(3.33/3.13)} \ge 4$, it follows that

$$4k^{2t(\frac{1}{h\nu}+1)}|\mathscr{C}(M,R)|^t \le M^t(10\eta)^{t/\eta_1}.$$

By (4.14), Lemma 4.1 and the induction hypothesis,

$$\begin{aligned} H_j(M) &\le \max\Bigg[(6\eta)^{tj/\eta_1}k^{\frac{4t(j-1)}{\eta_j'}}M^{tj+\frac{t(j-1)}{h}}, (10\eta)^{t/\eta_1}M^t \\ &\qquad \times \Bigg\{\sum_{M^{\frac{1}{h}}<q\le M^{\frac{1}{h}}R} H_{j-1}(M/q)^{\frac{1}{2t(j-1)}}\Bigg\}^{2t(j-1)}\Bigg] \\ &\le (10\eta)^{tj/\eta_1}\max\Bigg[k^{\frac{4t(j-1)}{\eta_j'}}M^{tj+\frac{t(j-1)}{h}}, C_{j-1} \\ &\qquad \times (e^2R)^{\frac{t}{2}(j-1)(j-2)}M^{2t(j-1)-\frac{t}{2}(h+k)+\Delta_{j-1}+t}S^{2t(j-1)}\Bigg], \end{aligned}$$

where

$$S = \sum_{M^{\frac{1}{h}}<q\le M^{\frac{1}{h}}R} q^E, \quad E = -1 + \frac{(t/2)(h+k)-\Delta_{j-1}}{2t(j-1)} = -1 + \frac{h(1-\alpha^{j-1})}{2j-2}.$$

Making use of the inequalities

$$1-\frac{\ell}{h} \le \alpha^\ell \le e^{-\ell/h} \le 1-\frac{\ell}{h}+\frac{\ell^2}{2h^2}, \tag{4.15}$$

it follows that $-\frac{5}{8} \le E \le -\frac{1}{2}$. Thus

$$S \le \int_1^{RM^{1/h}} x^E\,dx \le \frac{(RM^{1/h})^{E+1}}{E+1} \le \frac{8}{3}(RM^{1/h})^{E+1} \le eR^{1/2}M^{(1-\alpha^{j-1})/(2j-2)}.$$

We then obtain

$$H_j(M) \le (10\eta)^{tj/\eta_1} \max\left[k^{\frac{4t(j-1)}{\eta_j'}} M^{tj+\frac{t(j-1)}{h}}, C_{j-1}(e^2R)^{\frac{t}{2}(j^2-j)} M^{2tj-\frac{t}{2}(h+k)+\Delta_j}\right].$$

Write $f_j = j - \frac{j-1}{h} - h(1-\alpha^j)$, so that $f_1 = f_2 = 0$ and $f_j > 0$ for $j > 2$. Then

$$H_j(M) \le (10\eta)^{tj/\eta_1} M^{2tj-\frac{t}{2}(h+k)+\Delta_j} \max\left[k^{\frac{4t(j-1)}{\eta_j'}} M^{-tf_j}, C_{j-1}(e^2R)^{\frac{t}{2}(j^2-j)}\right].$$

By (4.15),

$$f_j \le j - \frac{j-1}{h} - h\left(\frac{j}{h} - \frac{j^2}{2h^2}\right) = \frac{j^2-2j+2}{2h} \le \frac{j^2-j}{2h} \quad (j \ge 2).$$

Since $M \ge M_j \ge R^{-h} P^{\alpha^{L-j}}$, we have

$$M^{-tf_j} \le R^{thf_j} P^{-tf_j\alpha^{L-j}} \le R^{\frac{t}{2}(j^2-j)} P^{-tf_j\alpha^{L-j}}.$$

Recalling the definition of E_j and η_j', we conclude that

$$H_j(M) \le (10\eta)^{tj/\eta_1} (e^2R)^{\frac{t}{2}(j^2-j)} M^{2tj-\frac{t}{2}(h+k)+\Delta_j} \max\left[e^{tE_j}, C_{j-1}\right].$$

Since $e^{tE_2} > C_1$, (4.13) follows at once. The Lemma then follows from (4.13) by taking $j = L$. $\square$

Lemma 4.3. *Suppose (4.9), (4.10) and (4.11) hold, and define* E_j *as in Lemma 4.2. Suppose that* $\log P \ge A$ *and*

$$x := \frac{4\log k}{A\eta\alpha} < 1. \tag{4.16}$$

Then

$$\max_{j\ge 2} E_j \le \frac{4\log k}{\eta}\left[1 + h\left(1 + \frac{(1-x)\log(1-x)}{x}\right)\right].$$

Proof. We have $E_j \le \max_{z\ge 2} F(z)$, where

$$F(z) = A(h - 1/h - \alpha x + \alpha z(x-1) - h\alpha^z).$$

By (4.16), $F(z) \to -\infty$ as $z \to \infty$ and $F(z)$ has a unique maximum point in $(-\infty,\infty)$. Solving $F'(y) = 0$, we see that

$$\alpha^y = \frac{\alpha(1-x)}{-h\log\alpha}. \tag{4.17}$$

If $y < 2$, then

$$\max_{z \geq 2} F(z) = F(2) = \frac{4 \log k}{\eta}$$

and the lemma follows in this case, because of the inequality $(1-x)\log(1-x) \geq -x$. Now assume $y \geq 2$. Since $-h \log \alpha = 1 + \frac{1}{2h} + \frac{1}{3h^2} + \cdots$, we have

$$\frac{1}{1 - \frac{1}{2h}} \leq -h \log \alpha \leq 1 + \frac{3h-1}{6h(h-1)} \leq 1 + \frac{1}{2h-2}.$$

Consequently, by (4.17)

$$x \geq \frac{1}{2h-1}. \tag{4.18}$$

Also,

$$\log(-h \log \alpha) \geq -\log\left(1 - \frac{1}{2h}\right) \geq \frac{1}{2h} + \frac{1}{8h^2}.$$

This gives

$$F(y) = A(h-1)(x + (1-x)V),$$

where

$$\begin{aligned} V &= 1 - \frac{1}{-h \log \alpha}\left(1 + \log(-h\log\alpha) - \log(1-x)\right) \\ &\leq 1 - \frac{6h(h-1)}{6h^2 - 3h - 1}\left(1 + \frac{1}{2h} + \frac{1}{8h^2}\right) + \frac{2h-2}{2h-1}\log(1-x) \\ &= \frac{5h+3}{4h(6h^2-3h-1)} + \frac{2h-2}{2h-1}\log(1-x) \\ &\leq \frac{1}{4h^2} + \frac{2h-2}{2h-1}\log(1-x). \end{aligned}$$

Using $(1-x)\log(1-x) \geq -x$ again, we obtain

$$\begin{aligned} F(y) &\leq \frac{(h-1)(1-x)A}{4h^2} + (h-1)Ax + \left(1 - \frac{1}{2h-1}\right)A(h-1)(1-x)\log(1-x) \\ &\leq \frac{(h-1)A}{4h^2} + (h-1)Ax\left(\frac{1}{2h-1} - \frac{1}{4h^2}\right) + (h-1)A(x + (1-x)\log(1-x)). \end{aligned}$$

By (4.18), we apply $1 \leq (2h-1)x$ in the first summand to obtain

$$\begin{aligned} F(y) &\leq (h-1)Ax\left(\frac{2h-1}{4h^2} + \frac{1}{2h-1} - \frac{1}{4h^2} + 1 + \frac{(1-x)\log(1-x)}{x}\right) \\ &\leq (h-1)Ax\left(\frac{1}{h} + 1 + \frac{(1-x)\log(1-x)}{x}\right). \end{aligned}$$

The lemma now follows from the definition of x (4.16). $\square$

Proof of Theorem 4. Let L be an integer, $2 \le L \le h/2$, and put $R = P^\eta$ and $A = Dk^2$. The hypotheses imply (4.9) and $\eta \le \frac{2}{3h}$. Next, by (1.10),

$$R \ge e^{\eta Dk^2} \ge k^{10} > \left(\frac{2}{\eta}\right)^3,$$

so (4.10) holds. Since $R \ge 6^{11}$, we may apply Lemma 2.2 with $\delta = \frac{1}{11}$. Suppose $Q = P^\omega$ with $\frac{1}{3} \le \omega \le 1$ and put $w = \lfloor 1.1\omega/\eta \rfloor$. Since $m! \le m^m$ and $(w+1)\eta \le 1.1\omega + \eta \le 1.2$,

$$|\mathscr{C}(Q,R)| \ge \frac{11^{-w}}{(w+1)w!}\frac{Q}{\log R} \ge \frac{1}{1.2}\left(\frac{1}{11w}\right)^w \frac{Q}{\log P} = Q^\beta,$$

where, by (1.10),

$$\begin{aligned} \beta &= 1 - \frac{\log(1.2\log P) + w\log(11w)}{\log Q} \\ &\ge 1 - \frac{3\log(1.2Dk^2)}{Dk^2} - \frac{1.1\log(12.1/\eta)}{\eta Dk^2} \\ &\ge 1 - 0.001 - 0.03 \ge 0.9. \end{aligned}$$

Thus, (4.11) holds and we may apply Lemmas 4.2 and 4.3. By (1.10), (4.16) and the bound $h \ge 54$,

$$x = \frac{4h\log k}{Dk^2\eta(h-1)} \in \left[\frac{18}{k}, 0.408\right],$$

so that

$$1 + \frac{(1-x)\log(1-x)}{x} = \frac{x}{2} + \frac{x^2}{6} + \frac{x^3}{12} + \cdots \le 0.5866x.$$

By Lemma 4.3,

$$\begin{aligned} \max_{j\ge 2} E_j &\le \frac{4\log k}{\eta}(1 + 0.5866hx) \\ &\le 2.57\frac{xk\log k}{\eta} \le 10.5\frac{\log^2 k}{Dk\eta^2}. \end{aligned}$$

Therefore, by Lemma 4.2,

$$J_{Lt,k,h}(\mathscr{C}(P,R)) \le C_L(e^2R)^{\frac{t}{2}L(L-1)}P^{2Lt-\frac{t}{2}(h+k)+\Delta_L},$$

where $\Delta_L = \frac{t(t-1)}{2} + ht\alpha^L$ and

$$\log C_L = \frac{10.5t\log^2 k}{Dk\eta^2} - tL\left(\left(\frac{1}{\eta} + h\right)\alpha^{L-1} - h\right)\log\left(\frac{1}{10\eta}\right).$$

By hypothesis, the number s satisfies $s = Lt + u$, where $0 \le u \le t$ and $2 \le L < L+1 \le h/2$. By Hölder's inequality,

(4.19)

$$\begin{aligned} J_{s,k,h}(\mathscr{C}(P,R)) &\le (J_{Lt,k,h}(\mathscr{C}(P,R)))^{1-u/t} (J_{Lt+t,k,h}(\mathscr{C}(P,R)))^{u/t} \\ &\le C_L^{1-u/t} C_{L+1}^{u/t} (e^2R)^{\frac{t}{2}L^2+L(u-t/2)} P^{2s-\frac{t}{2}(h+k)+(1-u/t)\Delta_L+(u/t)\Delta_{L+1}}. \end{aligned}$$

Next,

$$(1-u/t)\Delta_L + (u/t)\Delta_{L+1} = \frac{t(t-1)}{2} + ht\alpha^L\left(1-\tfrac{u}{ht}\right) < \frac{t(t-1)}{2} + hte^{-s/(ht)}$$

and

$$(e^2R)^{\frac{t}{2}L^2+L(u-t/2)} < (e^2R)^{s^2/(2t)} = e^{s^2/t}P^{\eta s^2/(2t)}.$$

For the constants, we use $\alpha^{L-1} > \alpha^L \ge \alpha^{s/t}$. Together with (4.19), this proves the theorem. □

5. Exponential Sums : Theorem 2 for large λ.

In this section, we apply Theorems 3 and 4 to prove Theorem 2 for large λ ($\lambda \ge 87$), using a variant of Vinogradov's method to relate $S(N,t)$ to both $J_{r,k}(P)$ and $J_{s,g,h}(\mathscr{B})$. Korobov's method [11] produces qualitatively similar bounds, but does not have the seperation of variables property (the c_i, d_i below in Lemma 5.1), and therefore one cannot easily modify it to incorporate incomplete systems (1.8). Rough calculations indicate that Korobov's method, when combined with Theorem 3, gives $S(N,t) \ll N^{1-1/(866\lambda^2)}$.

Lemma 5.1. *Suppose k, r and s are integers ≥ 2, and h and g are integers satisfying $1 \le h \le g \le k$. Let N be a positive integer, and M_1, M_2 be real numbers with $1 \le M_i \le N$. Let $\mathscr{B}$ be a nonempty subset of the positive integers $\le M_2$. Then*

$$S(N,t) \le 2M_1M_2 + \frac{t(M_1M_2)^{k+1}}{kN^k} + N\left(\frac{M_2}{|\mathscr{B}|}\right)^{\frac{1}{r}} \left((5r)^k M_2^{-2s} \lfloor M_1 \rfloor^{-2r+\frac{1}{2}k(k+1)} J_{r,k}(\lfloor M_1 \rfloor) J_{s,g,h}(\mathscr{B}) W_h \cdots W_g\right)^{\frac{1}{2rs}},$$

where

$$W_j = \min\left(2sM_2^j, \frac{2sM_2^j}{r\lfloor M_1\rfloor^j} + \frac{stM_2^j}{\pi j N^j} + \frac{4\pi j(2N)^j}{rt\lfloor M_1\rfloor^j} + 2\right) \qquad (j \ge 1).$$

Proof. For brevity write $M = \lfloor M_1 \rfloor$. For $N < R \le 2N$ and $0 < u \le 1$, we have

$$\begin{aligned}\left|\sum_{N<n\le R}(n+u)^{-it}\right| &= \frac{1}{M|\mathscr{B}|}\left|\sum_{\substack{a\le M_1\\ b\in\mathscr{B}}}\sum_{N<n+ab\le R}(n+ab+u)^{-it}\right| \\ &\le \frac{1}{M|\mathscr{B}|}\left|\sum_{\substack{a\le M_1\\ b\in\mathscr{B}}}\sum_{N<n\le R-1}(n+ab+u)^{-it}\right| + \frac{1}{M|\mathscr{B}|}\sum_{\substack{a\le M_1\\ b\in\mathscr{B}}}(2ab-1) \\ &\le \frac{N}{M|\mathscr{B}|}\max_{N\le z\le 2N}\left|\sum_{\substack{a\le M_1\\ b\in\mathscr{B}}} e^{-it\log(1+ab/z)}\right| + 2M_1M_2.\end{aligned}$$

For $0 \le x \le 1$ we have

$$\left|\log(1+x) - (x - x^2/2 + \cdots + (-1)^{k-1}x^k/k)\right| \le \frac{x^{k+1}}{k+1}. \tag{5.1}$$

Also $|e^{iy}-1| \le y$ for real y and $ab/z \le M_1M_2/N$. Thus, for some $z \in [N, 2N]$,

$$S(N,t) \le \frac{N}{M|\mathscr{B}|}|U| + \frac{t(M_1M_2)^{k+1}}{(k+1)N^k} + 2M_1M_2, \tag{5.2}$$

where $U = \sum_{a,b} e(\gamma_1(ab) + \cdots + \gamma_k(ab)^k)$ and $\gamma_j = (-1)^j t/(2\pi j z^j)$. By Hölder's inequality,

$$\begin{aligned}|U|^r &\le |\mathscr{B}|^{r-1} \sum_{b\in\mathscr{B}} \left| \sum_{a\le M_1} e(\gamma_1(ab) + \cdots + \gamma_k(ab)^k) \right|^r \\ &= |\mathscr{B}|^{r-1} \sum_{b\in\mathscr{B}} \varepsilon_b \left(\sum_{a\le M_1} e(\gamma_1(ab) + \cdots + \gamma_k(ab)^k) \right)^r \\ &= |\mathscr{B}|^{r-1} \sum_{b\in\mathscr{B}} \varepsilon_b \sum_{c_1,\dots,c_k} n(\boldsymbol{c}) e(\gamma_1 b c_1 + \cdots + \gamma_k b^k c_k),\end{aligned}$$

where ε_b are complex numbers with $|\varepsilon_b| = 1$, and for $\boldsymbol{c} = (c_1, \dots, c_k)$, $n(\boldsymbol{c})$ is the number of solutions of the simultaneous equations $c_j = a_1^j + \cdots + a_r^j$ $(1 \le j \le k)$ with each $a_i \in [1, M_1]$. A second application of Hölder's inequality gives

$$(5.3)\qquad \begin{aligned}|U|^{2rs} &\le |\mathscr{B}|^{2rs-2s} \left(\sum_{\boldsymbol{c}} n(\boldsymbol{c}) \right)^{2s-2} \left(\sum_{\boldsymbol{c}} n(\boldsymbol{c})^2 \right) T \\ &= |\mathscr{B}|^{2rs-2s} M^{2rs-2r} J_{r,k}(M) T,\end{aligned}$$

where

$$T = \sum_{\boldsymbol{c}} \left| \sum_{b\in\mathscr{B}} \varepsilon_b e(\gamma_1 b c_1 + \cdots + \gamma_k b^k c_k) \right|^{2s}.$$

For $0 < w \le \frac{1}{2}$, let $\ell(x; w) = \max(0, 1 - \frac{\|x\|}{w})$. This function has an absolutely and uniformly convergent Fourier series

$$\ell(x; w) = \frac{1}{\pi^2 w} \sum_{n=-\infty}^{\infty} \left(\frac{\sin \pi n w}{n} \right)^2 e(nx).$$

For $1 \le j \le k$ define

$$f_j(x) = \left(\frac{r M^j \sin(\pi x/(2rM^j))}{x} \right)^2,$$

and we note that $f_j(x) \ge 0$ for all x and $f_j(x) \ge 1$ for $1 \le x \le rM^j$. Since $1 \le c_j \le rM^j$ for each j, we have

$$\begin{aligned}T &\le \sum_{\substack{\boldsymbol{c} \\ -\infty < c_j < \infty}} \left| \sum_{b\in\mathscr{B}} \varepsilon_b e(\gamma_1 b c_1 + \cdots + \gamma_k b^k c_k) \right|^{2s} f_1(c_1) \cdots f_k(c_k) \\ &= \sum_{\substack{\boldsymbol{c} \\ -\infty < c_j < \infty}} \sum_{\substack{b_1,\dots,b_{2s} \\ b_i \in \mathscr{B}}} \varepsilon_{\boldsymbol{b}} e(\gamma_1 d_1 c_1 + \cdots + \gamma_k d_k c_k) f_1(c_1) \cdots f_k(c_k),\end{aligned}$$

where $|\varepsilon_{\boldsymbol b}|=1$ and $d_j=b_1^j+\cdots+b_s^j-b_{s+1}^j-\cdots-b_{2s}^j$ for $1\le j\le k$. For $\boldsymbol d=(d_1,\dots,d_k)$, write $J_{s,n,m}(\mathscr B;\boldsymbol d)$ for the number of $\boldsymbol b$ with $b_i\in\mathscr B$ for each i and $d_j=b_1^j+\cdots+b_s^j-b_{s+1}^j-\cdots-b_{2s}^j$ $(m\le j\le n)$. By Proposition ZRD, $J_{s,n,m}(\mathscr B;\boldsymbol d)\le J_{s,n,m}(\mathscr B)$. Then

$$\begin{aligned} T &\le \sum_{d_1,\dots,d_k} J_{s,k,1}(\mathscr B;\boldsymbol d)\left|\sum_{\boldsymbol c} e(\gamma_1 d_1 c_1+\cdots+\gamma_k d_k c_k) f_1(c_1)\cdots f_k(c_k)\right| \\ &= \sum_{\boldsymbol d} J_{s,k,1}(\mathscr B;\boldsymbol d)\prod_{j=1}^k \left|\sum_{c=-\infty}^{\infty} e(cd_j\gamma_j) f_j(c)\right| \\ &= \sum_{\boldsymbol d} J_{s,k,1}(\mathscr B;\boldsymbol d)\prod_{j=1}^k \left((rM^j)^2\frac{\pi^2}{2rM^j}\ell(d_j\gamma_j;\tfrac{1}{2rM^j})\right) \\ &= (\pi^2 r/2)^k M^{\frac12 k(k+1)}\sum_{\boldsymbol d} J_{s,k,1}(\mathscr B;\boldsymbol d)\prod_{j=1}^k \ell(d_j\gamma_j;\tfrac{1}{2rM^j}). \end{aligned}$$

Recalling the definition of $\ell(x;w)$, we obtain

$$T\le (5r)^k M^{\frac12 k(k+1)}\sum_{d_j\in\mathscr D_j\ \forall j} J_{s,k,1}(\mathscr B;\boldsymbol d), \tag{5.4}$$

where

$$\mathscr D_j=\{|d_j|<sM_2^j-1:\|d_j\gamma_j\|<\tfrac{1}{2rM^j}\}.$$

The sum in (5.4) may be interpreted as the number of solutions of the system of equations

$$\sum_{i=1}^s (x_i^j-y_i^j)=d_j\quad (1\le j\le k);\ \ x_i,y_i\in\mathscr B; d_j\in\mathscr D_j. \tag{5.5}$$

There are now several ways to proceed. A simple method is to ignore the equations in (5.5) corresponding to $j>g$ or $j<h$. Then, by Proposition ZRD, for each choice of $d_h,\dots,d_g$, the number of $\boldsymbol x,\boldsymbol y$ is $\le J_{s,g,h}(\mathscr B)$. Thus, by (5.4),

$$T\le (5r)^k M^{\frac12 k(k+1)} J_{s,g,h}(\mathscr B)\prod_{j=h}^g |\mathscr D_j|.$$

An alternate and slightly better method for bounding the number of solutions of (5.5) will be given in §8. Lastly, for positive δ, γ and K, we claim that

$$|\{|d|\le K:\|d\gamma\|<\delta\}|\le 4K\delta+2K\gamma+4\delta/\gamma+2. \tag{5.6}$$

Suppose that $\delta<1/2$, else (5.6) is trivial. The number of intervals of the form $[m-\delta,m+\delta]$ with integral m which intersect $[-K\gamma,K\gamma]$ is $\le 2\gamma K+1+2\delta\le 2\gamma K+2$.

Each such interval can contain at most $2\delta/\gamma + 1$ points of the form $d\gamma$, and this proves (5.6). Putting $K = sM_2^j - 1$, $\gamma = |\gamma_j|$ and $\delta = \frac{1}{2rM^j}$ gives $|\mathscr{D}_j| \le W_j$, hence

$$T \le J_{s,g,h}(\mathscr{B})(5r)^k M^{k(k+1)/2} W_h \cdots W_g.$$

Together with (5.2) and (5.3), this proves the lemma. □

Proof of Theorem 2 for $\lambda \ge 87$. Assume that

$$\lfloor M_1 \rfloor \ge M_2 \ge 100g, \quad s \le 2^g, \quad r \ge 13g, \quad r \ge s, \quad g \ge h \ge 3. \tag{5.7}$$

It turns out that the optimal parameters satisfy (5.7). By (5.7) and the definition of W_j,

$$W_j \le 4 + \frac{stM_2^j}{\pi N^j} + \frac{13g2^g N^j}{rtM_1^j} \le 2^{g+1} \max\left(1, \frac{tM_2^j}{N^j}, \frac{N^j}{tM_1^j}\right).$$

Suppose that

$$M_1 = N^{\mu_1}, \quad M_2 = N^{\mu_2}, \quad \mu_1 > \mu_2. \tag{5.8}$$

Then, the above bound for W_j is better than the trivial bound $2sM_2^j$ only when $\lambda < j < \lambda/(1 - \mu_1 - \mu_2)$. Let

$$\phi = g/\lambda, \quad \gamma = h/\lambda, \qquad 1 \le \gamma \le \frac{1}{1-\mu_2} < \frac{1}{1-\mu_1} \le \phi \le \frac{1}{1-\mu_1-\mu_2}. \tag{5.9}$$

We then have

$$W_h \cdots W_g \le 2^{g^2} M_2^{h+(h+1)+\cdots+g} N^{-H}, \tag{5.10}$$

where

$$H = \sum_{j=h}^{g} \min\left(j\mu_2, j - \lambda, \lambda - j(1 - \mu_1 - \mu_2)\right). \tag{5.11}$$

For $i = 1, 2$, write $\frac{\lambda}{1-\mu_i} = m_i + \beta_i$, where m_i is an integer and $0 \le \beta_i < 1$. Then

$$\begin{aligned} H &= \sum_{j=h}^{m_2}(j-\lambda) + \sum_{j=m_2+1}^{m_1} j\mu_2 + \sum_{j=m_1+1}^{g} (\lambda - j(1-\mu_1-\mu_2)) \\ &= \frac{(m_1^2+m_1)(1-\mu_1) + (m_2^2+m_2)(1-\mu_2) - h^2 + h - (1-\mu_1-\mu_2)(g^2+g)}{2} \\ &\quad + \lambda(h + g - m_1 - m_2 - 1) \\ &= \lambda^2\left(\phi + \gamma - \frac{\gamma^2}{2} - \frac{1-\mu_1-\mu_2}{2}\phi^2 - \frac{2-\mu_1-\mu_2}{2(1-\mu_1)(1-\mu_2)}\right) \\ &\quad + \lambda\left(\frac{\gamma}{2} - \frac{\phi}{2}(1-\mu_1-\mu_2)\right) - \frac{\beta_1(1-\beta_1)(1-\mu_1) + \beta_2(1-\beta_2)(1-\mu_2)}{2}. \end{aligned}$$

Since $\beta_i(1-\beta_i) \le \frac{1}{4}$,

$$
\begin{aligned}
H \ge \lambda^2 &\left(\phi + \gamma - \frac{\gamma^2}{2} - \frac{1-\mu_1-\mu_2}{2}\phi^2 - \frac{2-\mu_1-\mu_2}{2(1-\mu_1)(1-\mu_2)}\right) \\
&+ \lambda\left(\frac{\gamma}{2} - \frac{\phi}{2}(1-\mu_1-\mu_2)\right) - \frac{2-\mu_1-\mu_2}{8} \\
=&: H_2\lambda^2 + H_1\lambda - H_0.
\end{aligned} \tag{5.12}
$$

We shall take the near-optimal choice for the parameters

$$
\begin{aligned}
&\mu_1 = 0.1905, \quad \mu_2 = 0.1603, \quad k = \left\lfloor \tfrac{\lambda}{1-\mu_1-\mu_2} + 0.000003 \right\rfloor \ge 129, \\
&r = \lfloor \rho k^2 + 1 \rfloor, \qquad \rho \text{ taken from (1.7)},
\end{aligned} \tag{5.13}
$$

and approximate values (to be specified precisely later)

$$g \approx 1.2453\lambda, \qquad h \approx 1.1818\lambda, \qquad s \approx 0.3299h(t-1).$$

With these choices we quickly deduce that $S(N,t) \ll N^{1-1/(132.31\lambda^2)}$ for sufficiently large λ. By a standard argument (see §7), this implies (1.1) with $B = 4.42736$, but only for $1-\sigma$ sufficiently small. For completely explicit bounds, we pay more attention to the constants, sacrificing a little bit in B in order to get a fairly small value for A in Theorem 1.

By (5.13) and Theorem 3, we have

$$\lfloor M_1 \rfloor^{-2r+\frac{1}{2}k(k+1)} J_{r,k}(\lfloor M_1 \rfloor) \le C_1 M_1^{0.001k^2}, \tag{5.14}$$

where $C_1 = k^{\theta k^3}$ and θ is taken from (1.7). Let $Y = 300$ and assume that

$$N \ge e^{Y\lambda^2}, \tag{5.15}$$

for otheriwse trivially

$$S(N,t) \le N \le e^{Y/133.66} N^{1-1/(133.66\lambda^2)} \le 9.44 N^{1-1/(133.66\lambda^2)}.$$

We shall always choose g so that

$$106 \le g \le 1.254\lambda. \tag{5.16}$$

Thus by (5.13) and (5.15), $M_2 \ge e^{\mu_2 Y \lambda^2} \ge e^{0.1019 Y g^2}$. Let $D = 0.1019Y = 30.57$ and $\eta = \frac{1}{\xi g^{3/2}}$, where $3 \le \xi \le 6$. By (5.16), (1.10) holds and hence the hypotheses of Theorem 4 hold (with $P = M_2$ and $k = g$). By Theorem 4,

$$J_{s,g,h}(\mathscr{C}(M_2, M_2^{\eta})) \le C_2 P^{2s - \frac{t}{2}(h+g) + E_2}, \tag{5.17}$$

where
(5.18)

$$E_2 = \frac{1}{2}t(t-1) + \frac{\eta s^2}{2t} + ht\exp\{-\frac{s}{ht}\},$$
$$\log C_2 = \frac{s^2}{t} + \frac{10.5\xi^2 t g^2 \log^2 g}{D} - s\left((\xi g^{3/2}+h)(1-1/h)^{s/t} - h\right)\log(\xi g^{3/2}/10).$$

By (1.10) and (5.16),

$$R = M_2^\eta \ge e^{Dg^2\eta} \ge g^{10} > 6^{26}.$$

By Lemma 2.2 (with $\delta = \frac{1}{26}$) plus the inequality $w! \le (w/2.5)^w$ ($w \ge 50$), we have

$$\frac{M_2}{|\mathscr{C}(M_2,R)|} \le (\log R)(1.04\xi g^{3/2}+1)\left(\frac{27.04\xi g^{3/2}}{2.5}\right)^{1.04\xi g^{3/2}}$$
$$\le (\log N)C_3 \le C_3 N^{E_3},$$

where

(5.19)
$$C_3 = (10.82\xi g^{3/2})^{1.04\xi g^{3/2}},$$
$$E_3 = \frac{\log(Y\lambda^2)}{Y\lambda^2}.$$

By (5.13),

(5.20)
$$(5r)^k \le (40\lambda^2)^{1.6\lambda} \le \lambda^{5\lambda}$$

and

(5.21)
$$r \ge 7.509\lambda^2.$$

Consequently

$$\frac{E_3}{r} \le \frac{\log(Y\lambda^2)}{7.5Y\lambda^4}.$$

By Lemma 5.1, (5.10), (5.13), (5.14), (5.17) and (5.20), it follows that

(5.22)
$$S(N,t) \le \left(C_3^{\frac{1}{r}}\left(\lambda^{5\lambda}C_1C_2\right)^{\frac{1}{2rs}}\right)N^{1+E} + 2N^{0.36} + \frac{1}{k}N^{1-0.0000019476},$$
$$E = \frac{\log(Y\lambda^2)}{7.5Y\lambda^4} + \frac{1}{2rs}\left(-H + 0.001\mu_1 k^2 + \mu_2 E_2\right).$$

We also need bounds on k/λ, which by (5.13) can be written as

(5.23)
$$k_0 := \tfrac{1}{0.6492} - \tfrac{0.999997}{\lambda} \le \tfrac{k}{\lambda} \le \tfrac{1}{0.6492} + \tfrac{0.000003}{\lambda} =: k_1.$$

Lemma 5.2. *When $\lambda \ge 220$, we have*

$$S(N,t) \le 7.5 N^{1-1/(133.58\lambda^2)} \qquad (N \ge e^{300\lambda^2}).$$

Proof. We take

$$h = \lfloor 1.1818\lambda + \tfrac12 \rfloor, \quad g = \lfloor 1.2453\lambda + \tfrac12 \rfloor, \quad s = \lfloor \sigma h(t-1)+1 \rfloor, \quad \sigma = 0.3299. \tag{5.24}$$

By (5.9) and (5.24), (5.16) holds and also

$$|\gamma - 1.1818| \le \tfrac{1}{2\lambda}, \qquad |\phi - 1.2453| \le \tfrac{1}{2\lambda}. \tag{5.25}$$

Further, by (5.13) and (5.24),

$$g \ge 274, \quad h \ge 260, \quad t \ge 13, \quad k \ge 338, \quad s \ge 0.02294\lambda^2. \tag{5.26}$$

By (1.7), (5.13) and (5.14),

$$C_1 = k^{2.3291k^3} \le e^{9.2\lambda^3 \log \lambda}. \tag{5.27}$$

Taking

$$\xi = 6,$$

we have by (5.19) and (5.24),

$$C_3 \le e^{20.31\lambda^{3/2}\log\lambda}. \tag{5.28}$$

To bound C_2, we first note that by (5.24),

$$(1-1/h)^{s/t} \ge (1-1/h)^{\sigma(h-1)} \ge e^{-\sigma} \ge 0.71899.$$

This implies

$$(\xi g^{3/2} + h)(1-1/h)^{s/t} - h \ge 5.9785\lambda^{3/2} - 0.28101h \ge 5.956\lambda^{3/2}.$$

By (5.18), (5.24) and (5.26),

$$\begin{aligned} \log C_2 &\le 0.3907 s\lambda + 20.86 t\lambda^2 \log^2\lambda - 8.73 s\lambda^{3/2}\log\lambda \\ &\le 1.52\lambda^3\log^2\lambda - 8.72 s\lambda^{3/2}\log\lambda. \end{aligned} \tag{5.29}$$

By (5.21) and (5.26), $2rs \ge 0.3445\lambda^4$. Combining (5.21), (5.27), (5.28) and (5.29), we obtain

$$\begin{aligned} C_3^{\frac1r}\left(\lambda^{5\lambda}C_1C_2\right)^{\frac{1}{2rs}} &\le \exp\left\{ \frac{\log\lambda}{\lambda^{1/2}}\left(\frac{20.31 - 8.72/2}{7.509}\right) + \frac{\log\lambda}{0.3445}\left(\frac{5}{\lambda^3} + \frac{9.2 + 1.52\log\lambda}{\lambda}\right)\right\} \\ &\le e^{2.011} \le 7.48. \end{aligned} \tag{5.30}$$

By (5.22), it remains to bound E. Note that $-H + 0.001\mu_1 k^2 + \mu_2 E_2 < 0$. By (5.9), (5.13), (5.18) and (5.22),

$$\begin{aligned} E &\le \frac{\log(Y\lambda^2)}{7.5Y\lambda^4} + \frac{-H + 0.001\mu_1 k^2}{2.002\rho\sigma\gamma(\phi-\gamma)\lambda^2 k^2} + \frac{\mu_2 E_2}{2\rho k^2 s} \\ &\le \frac{1.52\times 10^{-7}}{\lambda^2} + \frac{-\lambda^2 H_2 - \lambda H_1 + H_0}{2.002\rho\sigma\gamma(\phi-\gamma)\lambda^2 k^2} + \frac{0.001\mu_1}{2.002\rho\sigma\gamma(\phi-\gamma)\lambda^2} \\ &\quad + \frac{\mu_2}{2\rho k^2}\left[\frac{\phi-\gamma+\frac{1}{\lambda}}{2\sigma\gamma} + \frac{\frac{t}{t-1}e^{-\sigma+\sigma/t}}{\sigma} + \frac{\sigma h g^{-3/2}}{12}\right]. \end{aligned}$$

By (5.26),

$$\tfrac{t}{t-1}e^{\sigma/t} \le 1 + \frac{1.33413}{t-1} = 1 + \frac{1.33413}{(\phi-\gamma)\lambda}.$$

Therefore

$$\lambda^2 E \le 1.52\times 10^{-7} + \frac{f(\gamma,\phi) + G_1/\lambda^{1/2} + G_2/\lambda}{\rho},$$

where, by (5.24) and (5.25),

$$\begin{aligned} f(\gamma,\phi) &= \frac{1}{2.002\sigma\gamma}\left[\frac{0.001\mu_1}{\phi-\gamma} + \frac{1}{k_1^2}\left(\frac{-H_2}{\phi-\gamma} + 1.001\mu_2\left(\frac{\phi-\gamma}{2} + \gamma e^{-\sigma}\right)\right)\right], \\ G_1 &= \frac{\mu_2\sigma\gamma\phi^{-3/2}}{24k_0^2} \le 0.0008, \\ G_2 &= \frac{1}{2.002\sigma(k/\lambda)^2}\left[\frac{-H_1 + H_0/\lambda + 1.33547\mu_2\gamma e^{-\sigma}}{\gamma(\phi-\gamma)} + \frac{1.001\mu_2}{2\gamma}\right]. \end{aligned}$$

Let U be the bracketed expression in the definition of G_2. By (5.12) (the definition of H_1 and H_0), (5.25) and (5.26),

$$\begin{aligned} U &\le \frac{0.3246\phi - 0.34608\gamma + 0.20615/\lambda}{\gamma(\phi-\gamma)} + \frac{1.001\mu_2}{2\gamma} \\ &= \frac{1.001\mu_2 + 0.6492}{2\gamma} + \frac{-0.02148 + \frac{0.20615}{h}}{\phi-\gamma} \\ &\le \frac{0.80967}{2.3636 - 1/\lambda} + \frac{-0.02148 + 0.20615/h}{0.0635 + 1/\lambda} \\ &\le 0.0392. \end{aligned}$$

Thus

$$G_2 \le \frac{0.0392}{2.002\sigma\gamma k_0^2} \le 0.021334.$$

Then

$$\begin{aligned} \lambda^2 E &\le 1.52\times 10^{-7} + \frac{f(\gamma,\phi) + 0.0008\lambda^{-1/2} + 0.021334\lambda^{-1}}{\rho} \\ &\le 0.00004711 + \frac{f(\gamma,\phi)}{\rho}. \end{aligned} \tag{5.31}$$

A short analysis with the aid of Maple shows that in the range $|\phi - 1.2453| \le \frac{1}{440}$, $|\gamma - 1.1818| \le \frac{1}{440}$, we have

$$f(\gamma,\phi) \le -0.0242145,$$

the maximum occuring at $\gamma = 1.1818 + \frac{1}{440}$, $\phi = 1.2453 - \frac{1}{440}$. By (1.7), (5.13) and (5.31), we conclude that

$$\lambda^2 E \le -0.0074862 \le -\frac{1}{133.58}.$$

Together with (5.22) and (5.30), this proves the lemma. □

Lemma 5.3. *When* $87 \le \lambda \le 220$, *we have*

$$S(N,t) \le 8.4 N^{1-1/(133.66\lambda^2)} \qquad (N \ge e^{300\lambda^2}).$$

Proof. Here we take

$$\xi = 3.6, \qquad s = \lfloor \sigma h t \rfloor + 1, \quad \sigma = 0.3299.$$

We choose g,h satisfying (5.16) and

$$g = \left\lfloor \tfrac{\lambda}{1-\mu_1} \right\rfloor + 1 + a, \quad h = \left\lfloor \tfrac{\lambda}{1-\mu_2} \right\rfloor - b, \quad t = g - h + 1, \quad a,b \in \{0,1\}.$$

To bound the exponent of N, consider $\lambda \in I = [\lambda_1, \lambda_2)$, a small interval on which each of the quantities $m_1 = \lfloor \frac{\lambda}{1-\mu_1} \rfloor$, $m_2 = \lfloor \frac{\lambda}{1-\mu_2} \rfloor$ and k (defined in (5.13)) is constant. We choose constant values of a and b in I, so that g, h, t, s, r are also fixed. By the definition of H, we have for $\lambda \in I$

$$\begin{aligned} H &= Z_0 + Z_1 \lambda, \\ Z_0 &= \frac{(m_1^2 + m_1)(1-\mu_1) + (m_2^2 + m_2)(1-\mu_2) - h^2 + h - (1-\mu_1-\mu_2)(g^2+g)}{2}, \\ Z_1 &= h + g - m_1 - m_2 - 1 = a - b \in \{-1,0,1\}. \end{aligned}$$

Therefore,

$$H \ge H' := Z_0 + \begin{cases} \lambda_1 & Z_1 = 1 \\ 0 & Z_1 = 0 \\ -\lambda_2 & Z_1 = -1 \end{cases}.$$

By (5.22),

$$E \le \frac{\log(Y\lambda_1^2)}{7.5 Y \lambda_1^4} - \frac{H' - 0.001\mu_1 k^2 - \mu_2 \left(\frac{t(t-1)}{2} + \frac{s^2}{\xi t g^{3/2}} + h t e^{-s/(ht)} \right)}{2rs} := E'.$$

Then, by (5.22), when $\lambda \in I$ we have

$$S(N,t) \le CN^{1-1/(u\lambda^2)} + \frac{1}{k} N^{1-1/(133\lambda^2)},$$

where $u = 1/(E'\lambda_1^2)$ and $C = C_3^{1/r}(\lambda^{5\lambda} C_1 C_2)^{1/(2rs)}$. A short computer program (Program 2 in the Appendix) is used to compute C and u in each interval, and to find the best choice for a and b (the choice which gives the smallest C subject to $u \le 133.66$). In all cases, $C \le 8.38$. For most λ, we take $b = 0$ and for $\lambda \in [136, 220]$ we take $a = 1$. This concludes the proof.

No choice of parameters g, h, s produced $C < 9.5$ in the range $86 \le \lambda \le 87$. □

Together, Lemma 5.2 and 5.3 prove Theorem 2 for $\lambda \ge 87$.

6. Theorem 2 for small λ

We begin with a general inequality derived from the Weyl shifting method. Suppose N is a positive integer and M is a real number satisfying $1 \le M \le N$. Arguing as in the proof of Lemma 5.1, for $N < R \le 2N$ and $0 < u \le 1$, we have

$$\left| \sum_{N<n\le R} (n+u)^{-it} \right| = \frac{1}{\lfloor M+1 \rfloor} \left| \sum_{m\le M+1} \sum_{N<n+m\le R} (n+m+u)^{-it} \right|$$
$$\le \frac{1}{M} \left| \sum_{m\le M} \sum_{N<n\le R-1} (n+m+u)^{-it} \right| + \frac{N}{M} + \frac{1}{M} \sum_{m\le M} (2m-1).$$

Therefore,

$$S(N,t) \le \frac{1}{M} \max_{0<u\le 1} \sum_{N<n\le 2N-1} \left| \sum_{m\le M} e^{-it\log(1+m/(n+u))} \right| + \frac{N}{M} + M. \tag{6.1}$$

Lemma 6.1. *If $|\alpha - p/q| \le 1/q^2$, $(p,q)=1$, m is a positive integer, and $x \ge 1$ and $y \ge 2$ are real numbers, then*

$$\sum_{n\le x} \min\left(y, \frac{1}{2\|\alpha m n\|} \right) \le \left(1 + \frac{2mx}{q} \right) \left(2q \log(ey) + 4y \right).$$

Proof. For $0 \le j \le 2q-1$ let I_j be the interval $[\frac{j}{2q}, \frac{j+1}{2q})$. The interval $[1,x]$ can be partitioned into intervals B_i, $1 \le i \le 1 + 2mx/q$, each of length $\le q/(2m)$. If $n, n' \in B_i$ and $\{\alpha m n\}, \{\alpha m n'\} \in I_j$ then

$$\left\| \frac{pm}{q}(n-n') \right\| \le \|\alpha m n - \alpha m n'\| + \left| \frac{m(n-n')}{q^2} \right| < \frac{1}{q},$$

hence $n = n'$. So, for $0 \le j \le q-1$, there are at most $G = 2 + 4mx/q$ values of n giving $\|\alpha m n\| \in I_j$. We take the summand to be y when $j \le q/y + 1$, thus

$$\sum_{n\le x} \min\left(y, \frac{1}{2\|\alpha m n\|} \right) \le Gy(q/y+2) + G \sum_{q/y+1<j\le q-1} q/j \le G(q+2y+q\log y). \qquad \square$$

Next, we use the Weyl method to prove Theorem 2 for $1 \le \lambda \le 2.6$. There is much room for improvement here, but the bounds below more than suffice for our purposes.

Lemma 6.2. *We have*

$$S(N,t) \le 5N^{1-1/20} \qquad (1 \le \lambda \le 1.9),$$
$$S(N,t) \le 30N^{1-1/83} \qquad (1.9 \le \lambda \le 2.6).$$

Consequently, when $1 \le \lambda \le 2.6$, *we have*

$$S(N,t) \le 1.81N^{1-1/(133\lambda^2)}.$$

Proof. Suppose $k \ge 2$. By (6.1) and (5.1), for some real number $z \in [N, 2N]$,

$$S(N,t) \le \frac{N}{M}|U| + \frac{N}{M} + M + \frac{tM^{k+1}}{(k+1)N^k}, \tag{6.2}$$

where

$$U = \sum_{m \le M} e^{-it((m/z)-m^2/(2z^2)+\cdots+(-1)^{k-1}m^k/(kz^k))}.$$

By the proof of Weyl's inequality (e.g. Lemma 2.4 of [27]), we have

$$|U|^{2^{k-1}} \le (2M)^{2^{k-1}-k} \sum_{\substack{h_1,\dots,h_{k-1} \\ |h_i| \le M-1}} \min\left(M, \frac{1}{2\|\alpha h_1 \cdots h_{k-1} k!\|}\right),$$

where $\alpha = t/(2\pi k z^k)$. There are at most $(k-1)(2M)^{k-2}$ vectors $(h_1, \cdots, h_{k-1})$ with some $h_i = 0$, thus

$$\begin{aligned} |U|^{2^{k-1}} \le (2M)^{2^{k-1}-k} \Bigg(&(k-1)M^{k-1}2^{k-2} \\ &+ 2^{k-1} \sum_{1 \le h \le M^{k-1}} d_{k-1}(h) \min\big(M, \frac{1}{2\|\alpha h k!\|}\big)\Bigg). \end{aligned} \tag{6.3}$$

Suppose $1 \le \lambda \le 1.9$. Let $q = \lfloor 1/\alpha \rfloor$ and note that $\frac{(4\pi-1)N^2}{t} \le q \le \frac{16\pi N^2}{t}$. Assume $M \ge 10000$. By (6.3) with $k = 2$ and Lemma 6.1,

$$\begin{aligned} |U|^2 &\le 9M + \frac{32M^2}{q} + (16M + 4q)\log(eM) \\ &\le \frac{32M^2 t}{(4\pi-1)N^2} + \left(17M + \frac{64\pi N^2}{t}\right)\log(eM). \end{aligned} \tag{6.4}$$

We may assume that $N \ge 5^{20}$, otherwise the claimed bound is trivial. We shall take $M = N^\mu$, where $\mu = \frac{2.95-\lambda}{3} \in [0.35, 0.65]$, so that $M \ge 5^7 > 10000$. By (6.2) and (6.4),

$$\begin{aligned}S(N,t) &\le N\left(\frac{32t}{(4\pi-1)N^2}+\left(\frac{17}{M}+\frac{64\pi N^2}{tM^2}\right)\log(eM)\right)^{1/2}+\frac{N}{M}+M+\frac{tM^3}{3N^2}\\ &\le N\left(3N^{\lambda-2}+\left(17N^{-0.35}+64\pi N^{-0.3}\right)\log N\right)^{1/2}+2N^{0.65}+\frac{N^{0.95}}{3}\\ &\le N\left(3N^{-0.1}+205N^{-0.3}\log N\right)^{1/2}+0.334N^{0.95}\\ &\le 4.1N^{0.95}.\end{aligned}$$

When $1.9\le\lambda\le 2.6$, we apply (6.3) with $k=3$, obtaining

$$|U|^4\le 2M\left(4M^2+4\sum_{1\le h\le M^2}d_2(h)\min\big(M,\frac{1}{2\|6\alpha h\|}\big)\right).$$

We shall use a crude upper bound on $\tau(h)$:

$$\begin{aligned}\frac{d_2(h)}{h^{1/3}}&=\prod_{p^e\|h}\frac{e+1}{p^{e/3}}\le\prod_p\max_{e\ge 0}\frac{e+1}{p^{e/3}}\\ &=\prod_{p\le 7}\max_{e\ge 0}\frac{e+1}{p^{e/3}}=\frac{24}{315^{1/3}}\le 3.53.\end{aligned}$$

Take $q=\lfloor\frac{6\pi z^3}{t}\rfloor$, so that $\frac{(6\pi-1)N^3}{t}\le q\le\frac{48\pi N^3}{t}$. By Lemma 6.1 (with $m=6$, $x=M^2$, $y=M$), we obtain

$$\begin{aligned}|U|^4&\le 8M^3+28.24M^{5/3}\sum_{1\le h\le M^2}\min\big(M,\frac{1}{2\|6\alpha h\|}\big)\\ &\le 8M^3+28.24M^{5/3}\left((2q+24M^2)\log(eM)+4M+\tfrac{48M^3}{q}\right).\end{aligned}\tag{6.5}$$

We choose $\mu=1-\frac{\lambda+1/50}{4}\in[0.345,0.52]$ and put $M=N^\mu$. Then

$$3-\lambda=3-4(1-\mu)+\tfrac{1}{50}=-\tfrac{49}{50}+4\mu\in\left[\mu+0.055,\mu(2+\tfrac{3}{26})\right],$$

and consequently

$$MN^{0.055}\le\frac{N^3}{t}\le M^{2+3/26}.$$

We assume that $N\ge 30^{60}$, otherwise the claimed bound is trivial. Then $M\ge 30^{20.7}$ and

$$N^{0.055}\ge 74000,\qquad \frac{\log(eM)}{M^{1/13}}\le 0.318.$$

Thus,

$$
\begin{aligned}
(2q+24M^2)\log(eM)+4M+\frac{48M^3}{q} &\le (96\pi M^{2+3/26}+24M^2)\log(eM) \\
&\qquad + 4M + \frac{48}{6\pi-1}\frac{M^2}{N^{0.055}} \\
&\le M^{2+5/26}\left((96\pi+24M^{-3/26})(0.318)+\frac{4}{M^{1+5/26}}+\frac{1}{26000M^{5/26}}\right) \\
&\le 96M^{2+5/26}.
\end{aligned}
$$

By (6.5), $|U|^4 \le 2712M^{4-\frac{11}{78}}$, and (6.2) then gives

$$S(N,t) \le 4N(7.22M^{-\frac{11}{312}}) + 2N^{0.655} + \tfrac{1}{4}N^{1-\frac{1}{50}} \le 30N^{1-1/83}.$$

This completes the proof of the first part of the lemma. The last part follows a general inequality: if λ is fixed and $0<d<c<1$, then

$$S(N,t) \le CN^{1-c} \quad (N\ge 1) \qquad \Longrightarrow \qquad S(N,t) \le C^{d/c}N^{1-d} \quad (N \ge 1). \tag{6.6}$$

For the proof, if $N \le C^{1/c}$, then trivially $S(N,t) \le N = N^dN^{1-d} \le C^{d/c}N^{1-d}$. When $N > C^{1/c}$, the hypothesis of (6.6) implies that

$$S(N,t) \le CN^{1-c} = CN^{d-c}N^{1-d} \le C\cdot C^{\frac{1}{c}(d-c)}N^{1-d} = C^{d/c}N^{1-d}.$$

For $\lambda\in[1,1.9]$, take $c=\frac{1}{20}$, $d=\frac{1}{133}$ in (6.6) and for $\lambda \in [1.9, 2.6]$ take $c=\frac{1}{83}$, $d=\frac{1}{133.66(1.9^2)}$. □

For larger λ, we relate $S(N,t)$ to $J_{s,k}(P)$ using an older method (§6.12 of [25]).

Lemma 6.3. *Suppose $k\ge 2$, $s \ge 2$, $N\ge 1$, $1 \le M \le Nt^{-\frac{1}{k+1}}$ and $t \le N^k$. Then*

$$S(N,t) \le \frac{4N^{1-\frac{1}{2s}}}{M}\left(\pi^k k! k^k W M^{\frac{1}{2}k(k+1)} J_{s,k}(M)\right)^{\frac{1}{2s}} + \frac{N}{M} + M,$$

where

$$W = \frac{2^{k+2}N^{k+1}}{k^2tM^k}+1.$$

Proof. By (6.1) and Hölder's inequality,

$$S(N,t) \le \max_{0<u\le 1} \frac{N^{1-\frac{1}{2s}}}{M}\left(\sum_{N<n\le 2N-1} |T(n)|^{2s}\right)^{\frac{1}{2s}} + \frac{N}{M}+M, \tag{6.7}$$

where

$$T(n) = \sum_{m\le M} e\left(-\frac{t}{2\pi}\log\left(1+\frac{m}{n+u}\right)\right).$$

With n fixed, let $\gamma_j = \gamma_j(n) = (-1)^j \frac{t}{2\pi j(n+u)^j}$ for $1 \le j \le k$. Define

$$S(x;\boldsymbol{\beta}) = \sum_{m \le x} e(m\beta_1 + \cdots + m^k\beta_k),$$
$$\delta(m;\boldsymbol{\beta}) = -\frac{t}{2\pi}\log\left(1+\frac{m}{n+u}\right) - \sum_{j=1}^k \beta_j m^j.$$

When $0 \le w \le M$,

$$\begin{aligned}|\delta'(w;\boldsymbol{\beta})| &\le \frac{tM^k}{2\pi N^{k+1}} + \sum_{j=1}^k j|\beta_j - \gamma_j|M^{j-1} \\ &\le \frac{1}{2\pi M} + \sum_{j=1}^k j|\beta_j-\gamma_j|M^{j-1}.\end{aligned} \tag{6.8}$$

Let Ω_n be the region $\{\boldsymbol{\beta} : |\beta_j - \gamma_j| \le \frac{1}{2\pi jkM^j}\ \forall j\}$. By (6.8), for $\boldsymbol{\beta} \in \Omega_n$ and $0 \le w \le M$, $|\delta'(w;\boldsymbol{\beta})| \le \frac{1}{\pi M}$. For any $\boldsymbol{\beta} \in \Omega_n$, partial summation gives

$$T(n) = S(M;\boldsymbol{\beta})e(\delta(\lfloor M\rfloor;\boldsymbol{\beta})) - 2\pi i\int_0^M S(w;\boldsymbol{\beta})e(\delta(w;\boldsymbol{\beta}))\delta'(w;\boldsymbol{\beta})\,dw,$$

and thus

$$|T(n)| \le |S(M;\boldsymbol{\beta})| + \frac{2}{M}\int_0^M |S(w;\boldsymbol{\beta})|\,dw =: S_0(\boldsymbol{\beta}).$$

Integrating over Ω_n then gives

$$|T(n)|^{2s} \le \frac{1}{|\Omega_n|}\int_{\Omega_n} S_0(\boldsymbol{\beta})^{2s}\,d\beta = \pi^k k! k^k M^{\frac12 k(k+1)}\int_{\Omega_n} S_0(\boldsymbol{\beta})^{2s}.$$

For any $\boldsymbol{\beta}$, the number of n with $\boldsymbol{\beta} \in \Omega_n$ is at most the number of n with $|\gamma_k(n) - \beta_k| \le \frac{1}{2\pi k^2 M^k}$. By hypothesis, $|\gamma_k(N) - \gamma_k(2N)| < \frac12$ and by the mean value theorem, when $N \le n \le 2N-2$,

$$|\gamma_k(n) - \gamma_k(n+1)| \ge \frac{t}{2\pi(2N)^{k+1}}.$$

Therefore the number of such is n is at most W. Hence

$$\sum_{N<n\le 2N-1} |T(n)|^{2s} \le \pi^k k! k^k M^{\frac12 k(k+1)} W \int_{\mathbb{U}^k} S_0(\boldsymbol{\beta})^{2s}\,d\boldsymbol{\beta}. \tag{6.9}$$

By Hölder's inequality,

$$\begin{aligned}S_0(\boldsymbol{\beta})^{2s} &\le 2^{2s-1}\left(|S(M;\boldsymbol{\beta})|^{2s} + \left(\frac{2}{M}\right)^{2s}\left(\int_0^M |S(w;\boldsymbol{\beta})|\,dw\right)^{2s}\right) \\ &\le 2^{2s-1}|S(M;\boldsymbol{\beta})|^{2s} + \frac{2^{4s-1}}{M}\int_0^M |S(w;\boldsymbol{\beta})|^{2s}\,dw.\end{aligned}$$

Thus

$$\begin{aligned}\int_{\mathbb{U}^k} S_0(\boldsymbol{\beta})^{2s}\,d\boldsymbol{\beta} &\le 2^{2s-1}\int_{\mathbb{U}^k}|S(M;\boldsymbol{\beta})|^{2s}\,d\boldsymbol{\beta} + \frac{2^{4s-1}}{M}\int_0^M\int_{\mathbb{U}^k}|S(w;\boldsymbol{\beta})|^{2s}\,d\boldsymbol{\beta}\,dw\\ &= 2^{2s-1}J_{s,k}(M) + \frac{2^{4s-1}}{M}\int_0^M J_{s,k}(w)\,dw\\ &\le 2^{4s}J_{s,k}(M).\end{aligned}$$

Combined with (6.7) and (6.9), this gives the lemma. □

Corollary 6.4. *Suppose $k \ge 4$, $1 \le n \le k^2$ and $s = nk$. Assume that*

$$J_{s,k}(P) \le CP^{2s-\frac{1}{2}k(k+1)+\Delta} \qquad (P \ge 1).$$

Then, for $t = N^\lambda$ with $k-1 \le \lambda \le k$, we have

$$S(N,t) \le 4N^{1-\frac{1}{2nk}}\left(C(2\pi k)^k k! N^{\frac{2+2\Delta}{k+1}}\right)^{\frac{1}{2nk}} + N^{\frac{k}{k+1}} + N^{\frac{1}{k+1}}.$$

Proof. This follows from Lemma 6.3, taking $\mu = 1 - \frac{\lambda}{k+1} \in [\frac{1}{k+1}, \frac{2}{k+1}]$, $M = N^\mu$ and noting that $W \le 2^k M$. □

Bounds for $J_{s,k}(P)$ with the best exponents of P come from Lemma 3.5, however the constants are very large. By using older methods without "repeat efficient differencing", we obtain bounds with far better constants, while sacrificing something in the exponents of P. In fact, using Corollary 6.4 with the older bounds for $J_{s,k}(P)$ gives

$$S(N,t) \le C_\lambda N^{1-1/16\lambda^2} \quad (6 \le \lambda \le 100),$$

which is far better than needed for Theorem 2. Since we will then use (6.6) to greatly reduce the constant (to $C_\lambda^{16/133.66}$), it is better for us to minimize C_λ rather than the exponent of P. Lemma 6.5 below comes from using Lemma 3.2 in a non-iterative way. For some s, even better constants can be obtained using an older variation of the method (Lemma 6.6), where solutions modulo a single prime are considered (as opposed to considering a set of k^3 primes).

Lemma 6.5. *Suppose $k \ge 4$ and $1 \le n \le k^2$. Suppose $0 < \omega \le \frac{1}{2}$ or $\omega = 1$, and let $\eta = 1 + \omega$. Put $V(\omega) = 6k^3 \log k$ if $\omega = 1$ and $V(\omega) = \max(e^{1.5+1.5/\omega}, \frac{18}{\omega}k^3 \log k)$ otherwise. If*

$$J_{nk,k}(P) \le CP^{2nk-\frac{1}{2}k(k+1)+\Delta} \qquad (P \ge 1),$$

then

$$J_{nk+k,k}(P) \le C' P^{2nk-\frac{1}{2}k(k+1)+\Delta'},$$

where $\Delta' = (1 - 1/k)\Delta$ and

$$C' = C \max\left[4k^3 k! \eta^{k^2-\Delta}, V(\omega)^\Delta\right].$$

Proof. This comes from Lemma 3.2 with $Q = P$, $M = P^{1/k}$, $r = k$, $d = 0$, $T = 1$, $s = nk$, $q = 1$ and $\psi_j(z) = z^j$ for each j. Lemma 2.1 implies that the interval $[P^{1/k}, \eta P^{1/k}]$ contains at least k^3 primes. Also, $P^{1-1/k} \ge k^{3k-3} \ge 32s^2$, so the hypotheses of Lemma 3.2 are satisfied. Together with the inequality

$$L_s(P, P/p; \mathbf{\Phi}; p, 1, k) \le P^k J_{s,k}(P/p),$$

this proves that for $P \ge V(\omega)^k$ and $k \le s \le k^3$, there is a prime $p \in [P^{1/k}, \eta P^{1/k}]$ giving

$$J_{s+k,k}(P) \le 4k^3 k! p^{2s+\frac{1}{2}(k^2-k)} P^k J_{s,k}(P/p). \tag{6.10}$$

The upper bound on p now gives the lemma. If $P < V(\omega)^k$, trivially

$$\begin{aligned} J_{(n+1)k,k}(P) &\le P^{2k} J_{nk,k}(P) \le C P^{2(n+1)k - \frac{1}{2}k(k+1) + \Delta} \\ &\le C V(\omega)^{k(\Delta - \Delta')} P^{2s - \frac{1}{2}k(k+1) + \Delta'}. \end{aligned}$$

□

Lemma 6.6. *Suppose $k \ge 9$, $k \le s \le k^3 - k$, $P \ge k^k$ and p is a prime in $[P^{1/k}, 2P^{1/k}]$. Then*

$$J_{s+k,k}(P) \le \max\left[(ep)^{2k-2}(k-1)^{2s+2} J_{s+k,k}(\tfrac{P}{p}), \tfrac{32}{k!}(s+k)^{2k} p^{2s+\frac{1}{2}(k^2-k)} P^k J_{s,k}(\tfrac{P}{p}) \right].$$

Proof. Let S_1 be the number of solutions of (1.4) (with $s \to s + k$) with at least k distinct residues modulo p among $x_1, \dots, x_{s+k}$ or at least k distinct residues modulo p among $y_1, \dots, y_{s+k}$. Let S_2 be the number of remaining solutions. Clearly $J_{s+k,k}(P) \le 2\max(S_1, S_2)$. Let

$$f(\boldsymbol{\alpha}; Q) = \sum_{x \le Q} e(\alpha_1 x + \cdots + \alpha_k x^k).$$

If $S_2 \ge S_1$, for $1 \le b \le p$ let

$$\begin{aligned} g(\boldsymbol{\alpha}; b) &= \sum_{\substack{x \le P \\ x \equiv b \pmod{p}}} e(\alpha_1 x + \cdots + \alpha_k x^k) \\ &= \sum_{1 \le y \le \frac{P+p-b}{p}} e(\alpha_1(py + b - p) + \cdots + \alpha_k(py + b - p)^k). \end{aligned}$$

Define $\mathscr{B}$ to be the set of $(b_1, \dots, b_{s+k})$ with $1 \le b_i \le p$ for each i, and containing at most $k-1$ distinct values. Then

$$|\mathscr{B}| \le \binom{p}{k-1}(k-1)^{k+s} \le \frac{p^{k-1}}{(k-1)!}(k-1)^{k+s} \le \tfrac{1}{2}(ep)^{k-1}(k-1)^{s+1}. \tag{6.11}$$

By Hölder's inequality,

$$\begin{aligned}
J_{s+k,k}(P) &\le 2\int_{\mathbb{U}^k}\left|\sum_{\boldsymbol{b}\in\mathscr{B}} g(\boldsymbol{\alpha};b_1)\cdots g(\boldsymbol{\alpha};b_{s+k})\right|^2 d\boldsymbol{\alpha}\\
&\le 2\int_{\mathbb{U}^k}\left(\sum_{\boldsymbol{b},\boldsymbol{b}'\in\mathscr{B}}|g(\boldsymbol{\alpha};b_1)|^{2s+2k}\right)^{\frac{1}{2s+2k}}\cdots\left(\sum_{\boldsymbol{b},\boldsymbol{b}'\in\mathscr{B}}|g(\boldsymbol{\alpha};b'_{s+k})|^{2s+2k}\right)^{\frac{1}{2s+2k}} d\boldsymbol{\alpha}\\
&= 2\sum_{\boldsymbol{b},\boldsymbol{b}'\in\mathscr{B}}\int_{\mathbb{U}^k}|g(\boldsymbol{\alpha};b_1)|^{2s+2k}\,d\boldsymbol{\alpha}.
\end{aligned}$$

By the binomial theorem, the last integral is $J_{s+k,k}(\frac{P+p-b_1}{p})$, hence

$$J_{s+k,k}(P) \le 2|\mathscr{B}|^2 J_{s+k,k}(P/p+1).$$

For brevity, write $P_1 = P/p+1$. We have $J_{s+k,k}(P_1) \le 2\max(S_3,S_4)$, where S_3 is the number of solutions of (1.4) with every $x_i, y_i \le P/p$ and S_4 is the number of remaining solutions. If $S_4 \ge S_3$, Hölder's inequality implies

$$\begin{aligned}
J_{s+k,k}(P_1) &\le 2(2s+2k)\int_{\mathbb{U}^k}|f(\boldsymbol{\alpha};P_1)|^{2s+2k-1}\,d\boldsymbol{\alpha}\\
&\le 4(s+k)\left(\int_{\mathbb{U}^k}|f(\boldsymbol{\alpha};P_1)|^{2s+2k}\,d\boldsymbol{\alpha}\right)^{1-\frac{1}{2s+2k}}\\
&= (4s+4k)J_{s+k,k}(P_1)^{1-\frac{1}{2s+2k}},
\end{aligned}$$

whence $J_{s+k,k}(P_1) \le (4s+4k)^{2s+2k}$. On the other hand, since $k\ge 9$ and $P \ge k^k$, counting only trivial solutions gives

$$J_{s+k,k}(P_1) \ge (P/p)^{s+k} \ge (\tfrac{1}{2}P^{1-1/k})^{s+k} > (4k^3)^{2s+2k} \ge (4s+4k)^{2s+2k},$$

a contradiction. Therefore, $J_{s+k,k}(P_1) \le 2J_{s+k,k}(P/p)$, and by (6.11),

$$J_{s+k,k}(P) \le 4|\mathscr{B}|^2 J_{s+k,k}(P/p) \le (ep)^{2k-2}(k-1)^{2s+2}J_{s+k,k}(P/p).$$

This proves the lemma in the case $S_2 \ge S_1$.

If $S_1 \ge S_2$, then S_1 is at most $2\binom{s+k}{k}$ times the number of solutions of (1.4) with $x_1,\cdots,x_k$ distinct modulo p. Let $\mathscr{X}$ be the set of k-tuples $(x_1,\cdots,x_k)$ which are distinct modulo p and

$$F(\boldsymbol{\alpha}) = \sum_{\boldsymbol{x}\in\mathscr{X}} e(\alpha_1(x_1+\cdots+x_k)+\cdots+\alpha_k(x_1^k+\cdots+x_k^k)).$$

Then, by the Cauchy-Schwarz inequality ,

$$\begin{aligned}J_{s+k,k}(P) &\le 2S_1 \le 4\binom{s+k}{k}\int_{\mathbb{U}^k} |F(\boldsymbol{\alpha})f(\boldsymbol{\alpha};P)^{2s+k}|\,d\boldsymbol{\alpha} \\ &\le 4\binom{s+k}{k}\left(\int_{\mathbb{U}^k} |F(\boldsymbol{\alpha})^2 f(\boldsymbol{\alpha};P)^{2s}|\,d\boldsymbol{\alpha}\right)^{\frac12}\left(\int_{\mathbb{U}^k} |f(\boldsymbol{\alpha};P)^{2s+2k}|\,d\boldsymbol{\alpha}\right)^{\frac12} \\ &= 4\binom{s+k}{k}\left(\int_{\mathbb{U}^k} |F(\boldsymbol{\alpha})^2 f(\boldsymbol{\alpha};P)^{2s}|\,d\boldsymbol{\alpha}\right)^{\frac12}(J_{s+k,k}(P))^{1/2}.\end{aligned}$$

Thus

$$J_{s+k,k}(P) \le 16\binom{s+k}{k}^2 \int_{\mathbb{U}^k} |F(\boldsymbol{\alpha})^2 f(\boldsymbol{\alpha};P)^{2s}|\,d\boldsymbol{\alpha} = 16\binom{s+k}{k}^2 S_3(p),$$

where $S_3(p)$ is defined as in the proof of Lemma 3.2 (with $\Psi_j(x) = x^j$ for $j = 1, \dots, k$). All the hypotheses of Lemma 3.2 hold, with $d = 0$, $T = 1$, $M = P^{1/k}$, $r = k$, $Q = P$ and $q = 1$. Recalling the definition (3.2) of $L_s(P, Q; \mathbf{\Psi}; p, q, r)$ and using (3.7),

$$\begin{aligned}J_{s+k,k}(P) &\le 16\frac{(s+k)^{2k}}{(k!)^2} 2k! p^{2s+\frac12 k(k-1)} L_s(P, P/p; \mathbf{\Phi}; p, 1, k) \\ &\le \tfrac{32}{k!}(s+k)^{2k} p^{2s+\frac12 k(k-1)} P^k J_{s,k}(P/p),\end{aligned}$$

and the lemma follows in the case $S_1 \ge S_2$. □

The chief advantage of Lemma 6.6 over Lemma 6.5 is the much smaller lower bound required for P (see (6.10)).

Lemma 6.7. *Suppose $k \ge 9$, $1 \le n \le k^2$ and*

$$J_{nk,k}(P) \le CP^{2nk-\frac12 k(k+1)+\Delta} \qquad (P \ge 1).$$

Suppose that $1 < \eta \le 2$ and that for $x \ge k$, there is a prime in $[x, \eta x]$. Then

$$J_{(n+1)k,k}(P) \le C' P^{2nk-\frac12 k(k+1)+\Delta'}, \tag{6.12}$$

where $\Delta' = (1 - 1/k)\Delta$ and

$$\begin{aligned}C' &= C\max\left[U^{\Delta}, \tfrac{32}{k!}(nk+k)^{2k}\eta^{k^2-\Delta}\right], \\ U &= \max\left[k, \left\{e^{2k-2}(k-1)^{2kn+2}\right\}^{\frac{1}{2nk-k(k+1)/2+\Delta'+2}}\right].\end{aligned}$$

Proof. If $P \le U^k$ then as in the proof of Lemma 6.5, we have the trivial estimate

$$J_{(n+1)k,k}(P) \le CU^{k(\Delta-\Delta')} P^{2(n+1)k-\frac12 k(k+1)+\Delta'}.$$

Next suppose $P > U^k \ge k^k$. We prove (6.12) by induction on $\lfloor P \rfloor$, observing that (6.12) for integral $P = m$ implies (6.12) for $m \le P < m+1$. Assume (6.12) is true for $\lfloor P \rfloor \le Q-1$, where Q is an integer $\ge U^k$, and apply Lemma 6.6. If the first term in the maximum in the conclusion on Lemma 6.6 is largest, (6.12) follows from the bound $p \ge U$ and the induction hypothesis on m. If the second term in the maximum is largest, (6.12) follows from the upper bound $p \le \eta P^{1/k}$ and the upper bound on $J_{rk,k}(P)$. □

Lemma 6.8. *Theorem 2 holds for* $2.6 \le \lambda \le 87$. *In particular, for each row of Table 6.1, when* λ *is in the stated range,*

$$S(N,t) \le CN^{1-1/(133.66\lambda^2)} \quad (N \ge 1).$$

Proof. Take k, n and n_0 from a row of the table. For reasons connected with the size of U in Lemma 6.7, it is advantageous to use a completely trivial bound

$$J_{nk,k}(P) \le P^{2nk-2k} J_{k,k}(P) \le k! P^{2nk-\frac{1}{2}k(k+1)+\Delta_n}, \quad \Delta_n = \frac{1}{2}k^2(1-1/k)$$

for $1 \le n \le n_0$. We then proceed iteratively, taking a bound of the form

$$J_{nk,k}(P) \le C_n P^{2nk-\frac{1}{2}k(k+1)+\Delta_n} \qquad (P \ge 1)$$

and producing a bound

$$J_{(n+1)k,k}(P) \le C_{n+1} P^{2nk-\frac{1}{2}k(k+1)+\Delta_{n+1}} \qquad (P \ge 1),$$

where $\Delta_{n+1} = (1-1/k)\Delta_n$ and C_{n+1} is the smaller of the constants coming from Lemmas 6.7 (only for $k \ge 9$) or 6.5 (with optimal choice of ω). As for the number η in Lemma 6.7, (2.1) implies that

$$\begin{aligned} \pi(1.12x) - \pi(x) &\ge \frac{x}{\log x}\left[1.12\left(1 + \frac{1/2 - \log 1.12}{\log x}\right) - 1 - \frac{3}{2\log x}\right] \\ &\ge \frac{x}{\log x}\left(0.12 - \frac{1.067}{\log x}\right) > 0 \quad (x \ge 7300). \end{aligned}$$

Using a table of primes < 7300, we find that the following are admissible choices for η:

$$\eta = \begin{cases} 17/13 & 9 \le k \le 13 \\ 29/23 & 14 \le k \le 32 \\ 53/47 & k \ge 33. \end{cases}$$

The optimal value of ω in Lemma 6.5 is found by solving

$$4k^3 k!(1+\omega)^{k^2-\Delta_n} = \max(e^{1.5+1.5/\omega}, \tfrac{18}{\omega}k^3 \log k), \tag{6.13}$$

λ	k	n_0	n	C	λ	k	n_0	n	C
2.6–4	4	1	13	2.5543	45–46	46	44	365	3.5897
4–5	5	1	17	1.7474	46–47	47	46	375	3.6728
5–6	6	1	22	1.7805	47–48	48	48	386	3.7580
6–7	7	1	28	1.8406	48–49	49	50	397	3.8453
7–8	8	1	34	1.9173	49–50	50	52	408	3.9348
8–9	9	3	40	1.6808	50–51	51	54	419	4.0266
9–10	10	3	46	1.7062	51–52	52	56	430	4.1207
10–11	11	3	52	1.7362	52–53	53	58	441	4.2171
11–12	12	4	59	1.7678	53–54	54	60	452	4.3160
12–13	13	4	66	1.8021	54–55	55	63	465	4.4174
13–14	14	5	73	1.8295	55–56	56	65	476	4.5214
14–15	15	6	81	1.8669	56–57	57	67	487	4.6280
15–16	16	6	88	1.9057	57–58	58	69	498	4.7373
16–17	17	7	96	1.9464	58–59	59	71	509	4.8494
17–18	18	8	104	1.9883	59–60	60	74	522	4.9643
18–19	19	8	111	2.0317	60–61	61	76	533	5.0821
19–20	20	9	119	2.0766	61–62	62	79	546	5.2030
20–21	21	10	127	2.1229	62–63	63	81	557	5.3268
21–22	22	11	136	2.1706	63–64	64	84	569	5.4539
22–23	23	11	143	2.2190	64–65	65	86	581	5.5841
23–24	24	12	152	2.2688	65–66	66	89	593	5.7176
24–25	25	13	161	2.3201	66–67	67	91	605	5.8546
25–26	26	14	169	2.3728	67–68	68	94	617	5.9950
26–27	27	15	178	2.4270	68–69	69	96	629	6.1390
27–28	28	17	188	2.4826	69–70	70	99	642	6.2867
28–29	29	17	196	2.5398	70–71	71	102	654	6.4381
29–30	30	19	206	2.5987	71–72	72	104	666	6.5934
30–31	31	20	215	2.6590	72–73	73	107	679	6.7527
31–32	32	21	224	2.7210	73–74	74	110	691	6.9160
32–33	33	23	233	2.6797	74–75	75	113	704	7.0836
33–34	34	25	243	2.7396	75–76	76	116	717	7.2553
34–35	35	26	252	2.8010	76–77	77	118	729	7.4315
35–36	36	28	263	2.8641	77–78	78	121	742	7.6122
36–37	37	29	272	2.9287	78–79	79	124	754	7.7975
37–38	38	31	283	2.9950	79–80	80	127	767	7.9876
38–39	39	32	292	3.0630	80–81	81	130	780	8.1825
39–40	40	34	303	3.1327	81–82	82	133	793	8.3825
40–41	41	36	313	3.2042	82–83	83	136	806	8.5876
41–42	42	37	323	3.2775	83–84	84	139	819	8.7979
42–43	43	39	333	3.3526	84–85	85	143	833	9.0136
43–44	44	41	344	3.4297	85–86	86	146	846	9.2350
44–45	45	43	355	3.5088	86–87	87	149	859	9.4620

TABLE 6.1

obtaining a positive real solution ω_0. The solution is unique since the left side of (6.13) is increasing in ω, while the right side is decreasing. If $\omega_0 \ge 1$, we take $\omega = 1$. If $\frac{1}{2} \le \omega_0 < 1$ we take ω to be either $\frac{1}{2}$ or 1, whichever gives the best constant C'. Otherwise take $\omega = \omega_0$.

Having computed admissible sequences C_n and Δ_n, we turn to Lemma 6.3 and Corollary 6.4 to bound $S(N, t)$. When $2.6 \le \lambda \le 4$ ($k = 4$, $n = 12$), take $\mu = 1 - \frac{\lambda}{5}$,

$M = N^{\mu}$ and apply Lemma 6.3. We have $W \le 2^k M$ and thus

$$S(N,t) \le 4\left(4!(8\pi)^4 C_n\right)^{\frac{1}{2nk}} N^{1-c} + 2N^{0.8}, \quad c = \frac{1}{2nk}(1 - 0.48(1+\Delta)) < 0.8.$$

Hence

$$S(N,t) \le \left(4\left(4!(8\pi)^4 C_n\right)^{\frac{1}{2nk}} + 2\right) N^{1-c} \quad (N \ge 1).$$

Applying (6.6) then gives the claimed inequality. For $\lambda \ge 4$, we use Corollary 6.4 directly, obtaining

$$S(N,t) \le \left(4\left(k!(2\pi k)^k C_n\right)^{\frac{1}{2nk}} + 2\right) N^{1-c}.$$

Then (6.6) implies the stated claim. A short computer program (Program 3 in the appendix) provided the computations of C_n and Δ_n, and found the best choices of parameters n_0 and n. The values of C listed in the table have been rounded up in the last displayed decimal place. □

7. Bounding $\zeta(s)$ and $\zeta(s,u)$

We start with a crude bound for $\zeta(s)$ and $\zeta(s,u)$ which takes care of s with either σ or t small.

Lemma 7.1. *Suppose $\frac12 \le \sigma \le 1$, $0 < u \le 1$, $t \ge 3$ and $s = \sigma + it$. If either $\sigma \le \frac{15}{16}$ or $t \le 10^{100}$, then*

$$|\zeta(s)|, |\zeta(s,u) - u^{-s}| \le 58.1 t^{4(1-\sigma)^{3/2}} \log^{2/3} t.$$

Proof. Applying integration by parts, when $\sigma > 0$ we have

$$\zeta(s,u) = \sum_{0\le n\le N} \frac{1}{(n+u)^s} + \frac{(N+\frac12+u)^{1-s}}{s-1} + s\int_{N+1/2}^\infty \frac{1/2-\{w\}}{(w+u)^{s+1}}\,dw, \tag{7.1}$$

where N is a positive integer. We take $N = \lfloor t \rfloor$, and note that $\frac{d^2}{dn^2}(n+u)^{-\sigma} > 0$. Therefore,

$$\begin{aligned}
|\zeta(s,u) - u^{-s}| &\le \int_{1/2+u}^{N+1/2+u} \frac{dw}{w^\sigma} + \frac{(N+\frac12+u)^{1-\sigma}}{t} + \frac{|s|}{2}\int_{N+1/2}^\infty \frac{dw}{(w+u)^{1+\sigma}} \\
&= \int_{1/2+u}^{N+1/2+u} \frac{dw}{w^\sigma} + \frac{(N+\frac12+u)^{1-\sigma}}{t} + \frac{|s|(N+\frac12+u)^{-\sigma}}{2\sigma} \\
&\le \int_{1/2+u}^{N+1/2+u} \frac{dw}{w^\sigma} + (1+\tfrac1t)(t+3/2)^{1-\sigma}.
\end{aligned}$$

If $\sigma < 1$,

$$\int_{1/2+u}^{N+1/2+u} \frac{dw}{w^\sigma} \le \frac{(N+1/2+u)^{1-\sigma}}{1-\sigma} \le \frac{(t+3/2)^{1-\sigma}}{1-\sigma}$$

and for all $\sigma \in (0,1]$, we have

$$\int_{1/2+u}^{N+1/2+u} \frac{dw}{w^\sigma} \le (N+1/2+u)^{1-\sigma}\int_{1/2+u}^{N+1/2+u} \frac{dw}{w} \le (t+3/2)^{1-\sigma}\log(2N+1).$$

Therefore, we obtain the inequality

$$|\zeta(s,u) - u^{-s}| \le (t+3/2)^{1-\sigma}\left(1 + \tfrac1t + \min\left(\tfrac{1}{1-\sigma}, \log(2t+1)\right)\right). \tag{7.2}$$

Consider first the case when $t \ge 3$ and $\frac12 \le \sigma \le \frac{15}{16}$. Here $(1-\sigma) \le 4(1-\sigma)^{3/2}$, so by (7.2)

$$|\zeta(s,u) - u^{-s}| \le \sqrt{1.5}\, t^{4(1-\sigma)^{3/2}}\left(\tfrac43 + 16\right) \le 21.3 t^{4(1-\sigma)^{3/2}}.$$

Next, if $\frac{15}{16} \le \sigma \le 1$ and $3 \le t \le 10^{100}$, (7.2) gives

$$|\zeta(s,u) - u^{-s}| \le (t+3/2)^{1-\sigma}(1 + 1/t + \log(2t+1)).$$

If $3 \le t \le 10^6$, the right side is ≤ 36.8. If $t > 10^6$, the right side is

$$\le 1.123t^{1-\sigma}\log t = 1.123\left(t^{4(1-\sigma)^{3/2}}\log^{2/3} t\right)\left(t^{1-\sigma-4(1-\sigma)^{3/2}}\log^{1/3} t\right).$$

The maximum of $1-\sigma-4(1-\sigma)^{3/2}$ is $\frac{1}{108}$, thus

$$|\zeta(s,u) - u^{-s}| \le 58.1t^{4(1-\sigma)^{3/2}}\log^{2/3} t.$$

Lastly, taking $u \to 0^+$ shows that the lemma holds for $|\zeta(s)|$ as well. □

Lemma 7.2. *If $s = \sigma + it$, $\frac{15}{16} \le \sigma \le 1$, $t \ge 10^{100}$ and $0 < u \le 1$, then*

$$\left|\zeta(s,u) - \sum_{0\le n\le t}(n+u)^{-s}\right| \le 10^{-80}.$$

Proof. Let $E(s,u) = \zeta(s,u) - \sum_{0\le n\le t}(n+u)^{-s}$. By (7.1) with $N = \lfloor t\rfloor$,

$$\begin{aligned}|E(s,u)| &\le \frac{(t+3/2)^{1-\sigma}}{t} + |s|\left|\int_{\lfloor t\rfloor+1/2+u}^{\infty}\frac{1/2-\{w\}}{w^{1+s}}\,dw\right| \\ &\le \frac{(t+3/2)^{1-\sigma}}{t} + \frac{3|s|}{4(t-1/2)^{\sigma+1}} + |s|\left|\int_t^{t^2}\frac{1/2-\{w\}}{w^{s+1}}\,dw\right| + \frac{|s|t^{-2\sigma}}{2\sigma} \\ &\le 10^{-81} + (t+1)\left|\int_t^{t^2}\frac{\{w\}-1/2}{w^{\sigma+1}}(\cos(t\log w) - i\sin(t\log w))\,dw\right|.\end{aligned}$$

We bound the intergal using the Fourier expansion $\{x\} - \frac{1}{2} = -\frac{1}{\pi}\sum_{m=1}^{\infty}\frac{\sin(2\pi mx)}{m}$, as in [3]. We also use the trigonometric identities

$$\sin a\sin b = \frac{\cos(a-b)-\cos(a+b)}{2}, \qquad \sin a\cos b = \frac{\sin(a+b)+\sin(a-b)}{2}.$$

Therefore, writing

$$I_m = \max_{h=\sin,\cos}\left|\int_t^{t^2}\frac{h(t\log x + 2\pi mx)}{x^{1+\sigma}}\,dx\right| + \left|\int_t^{t^2}\frac{h(t\log x - 2\pi mx)}{x^{1+\sigma}}\,dx\right|$$

and separating real and imaginary parts, we obtain

$$|E(s,u)| \le 10^{-81} + \frac{t+1}{\pi}\sum_{m=1}^{\infty}\frac{I_m}{m}. \tag{7.3}$$

To bound I_m, let $f(x) = x^{-\sigma}/(t \pm 2\pi mx))$ and $g(x) = k(t\log x \pm 2\pi mx)$, where $k'(x) = h(x)$ and $k(x) \in \{\pm\sin(x), \pm\cos(x)\})$. Since f is monotonic on $[t, t^2]$, we obtain

$$\begin{aligned}\left|\int_t^{t^2} \frac{h(t\log x \pm 2\pi mx)}{x^{1+\sigma}}\,dx\right| &= \left|\int_t^{t^2} f(x)g'(x)\,dx\right| \\ &= \left|f(t^2)g(t^2) - f(t)g(t) - \int_t^{t^2} g(x)f'(x)\,dx\right| \\ &\le |f(t)g(t)| + |f(t^2)g(t^2)| + \max_{t\le x\le t^2}|g(x)|\int_t^{t^2}|f'(x)|\,dx \\ &= |f(t)g(t)| + |f(t^2)g(t^2)| + \max_{t\le x\le t^2}|g(x)||f(t^2)-f(t)| \\ &\le \frac{4}{t^{1+\sigma}(2\pi m \pm 1)}.\end{aligned}$$

Therefore,

$$I_m \le \frac{4}{t^{1+\sigma}(2\pi m+1)} + \frac{4}{t^{1+\sigma}(2\pi m-1)} = \frac{16\pi m}{t^{1+\sigma}(4\pi^2m^2-1)} \le \frac{16\pi}{(4\pi^2-1)t^{1+\sigma}m}.$$

Together with (7.3), this proves the lemma. $\square$

Lemma 7.3. *Suppose that* $S(N,t) \le CN^{1-1/(D\lambda^2)}$ $(1 \le N \le t)$ *for positive constants* C *and* D*, where* $\lambda = \frac{\log t}{\log N}$*. Let* $B = \frac{2}{9}\sqrt{3D}$*. Then, for* $\frac{15}{16} \le \sigma \le 1$*,* $t \ge 10^{100}$ *and* $0 < u \le 1$*, we have*

$$|\zeta(s)| \le \left(\frac{C+1+10^{-80}}{\log^{2/3} t} + 1.569CD^{1/3}\right) t^{B(1-\sigma)^{3/2}}\log^{2/3} t,$$

$$|\zeta(s,u) - u^{-s}| \le \left(\frac{C+1+10^{-80}}{\log^{2/3} t} + 1.569CD^{1/3}\right) t^{B(1-\sigma)^{3/2}}\log^{2/3} t.$$

Proof. Let

$$S_1(u) = \sum_{1\le n\le t}(n+u)^{-s}.$$

By Lemma 7.2, $|\zeta(s,u) - u^{-s}| \le 10^{-80} + S_1(u)$. Put $r = \lceil \frac{\log t}{\log 2}\rceil$. By partial summation,

$$\begin{aligned}|S_1(u)| &\le 1 + \sum_{j=0}^{r-1}\left|\sum_{2^j<n\le\min(t,2^{j+1})}(n+u)^{-\sigma-it}\right| \\ &\le 1 + \sum_{j=0}^{r-1}(2^j)^{-\sigma}S(2^j,t) \\ &\le 1 + C\sum_{j=0}^{r-1}e^{g(j)},\end{aligned}$$

where

$$g(j) = (1-\sigma)(j\log 2) - \frac{(j\log 2)^3}{D\log^2 t}.$$

As a function of x, $g(x)$ is increasing on $[0, x_0]$ and decreasing on $[x_0, \infty)$, where $x_0 \log 2 = \sqrt{D(1-\sigma)/3}\log t$. Thus

$$\begin{aligned}\frac{|S_1(u)|-1}{C} &\le e^{g(x_0)} + \int_0^r e^{g(x)}\,dx \\ &\le t^{B(1-\sigma)^{3/2}} + \frac{D^{1/3}\log^{2/3} t}{\log 2}\int_0^\infty e^{3y^2u-u^3}\,du,\end{aligned}$$

where $y = \sqrt{(1-\sigma)/3}D^{1/6}\log^{1/3} t$. To bound the last integral, we make use of the inequality

$$e^{-2y^3}\int_0^\infty e^{3y^2u-u^3}\,du \le 1.0875034 \qquad (y \ge 0),$$

where the maximum occurs near $y = 0.710$. Therefore

$$\frac{|S_1(u)|-1}{C} \le t^{B(1-\sigma)^{3/2}}\left(1 + \frac{1.0875034}{\log 2}D^{1/3}\log^{2/3} t\right),$$

which proves the lemma. □

Proof of Theorem 1. Apply Lemma 7.3 using $C = 9.463$, $D = 133.66$ (from Theorem 2). □

8. Possible improvements to the constant B

There are a number of ways in which the constant B in Theorem 1 may be improved, and we sketch three of them below. To provide complete details would involve a substantial lengthening of this paper, and even more work would be required to obtain a decent constant A. Taken together, the three ideas have the potential to reduce the constant B only to about 4.1.

1. As noted in section 3, there are some improvements possible in the method for bounding $J_{s,k}(P)$. Tyrina's method could be used for small s (when $\Delta \ge \frac{4}{9}k^2$), and in Lemma 3.5 we could take $r \approx \sqrt{k^2+k-2\Delta}$ in Lemma 3.5. The end result is a slight reduction in the constant $\frac{3}{8}$ appearing in the definition of Δ_s in Theorem 3. This can lower B by less than 0.02.

2. As mentioned in section 4, the use of repeat efficient differencing (repeatedly forming divided differences of the polynomials Ψ_j as in [34]) produces superior bounds for $J_{s,g,h}(\mathscr{C}(P,R))$. Preliminary computations indicate a potential reduction in B of $4-5\%$, or 0.2 at most, making it hardly worth the effort of working out the details. There is also the problem of obtaining good explicit constants (e.g. e^C in Theorem 4). In particular, when Wooley's methods are used directly, the constants C are far too large to be of any use in bounding $\zeta(s)$. Referring to Lemma 4.1 of [34], relations (4.9) and (4.10) essentially bound $J_{s,g,h}$ in terms of $J_{s-1,g,h}$. When iterated, the constants grow too rapidly with s. In our Lemma 4.1 above, we avoided this pitfall by an application of Hölder's inequality at the end of the third case (assuming $S_3 = \max(S_1,S_2,S_3,S_4)$), a tool which is unavailable when using repeat efficient differencing. Incidentally, this idea was also used in the proof of Lemma 6.7 above. Presumably some clever argument would overcome this problem.

3. In the estimation of the quantity T in section 5, the number of solutions of (5.5) may be bounded in a more sophisticated way. First we note that when $sM_2^j|\gamma_j| \le \frac14$ (essentially $j \ge \frac{\lambda}{1-\mu_2}$), $\mathscr{D}_j$ is the set of integers in an interval of the form $[-D_j, D_j]$, where D_j is a non-negative integer. If in addition $|\gamma_j| \ge \frac{1}{2rM^j}$ (essentially $\frac{\lambda}{1-\mu_2} \le j \le \frac{\lambda}{1-\mu_1}$), in fact $\mathscr{D}_j = \{0\}$ (i.e. $D_j = 0$ in this case).

Let h_0 be the smallest integer with $D_{h_0} = 0$ and let $\tilde g$ be the largest integer with $|\gamma_{\tilde g}| \ge \frac{1}{2rM^{\tilde g}}$. Assuming $h_0 \le h \le \tilde g \le g \le k$, the number of solutions of (5.5) is at most $J^*_{s,g,h}(\mathscr{B};\boldsymbol{D})$, the number of solutions of

$$\sum_{i=1}^{s}(x_i^j - y_i^j) = d_j \quad (h \le j \le g), \tag{8.1}$$

with $x_i, y_i \in \mathscr{B}$ and $|d_j| \le D_j$ for each j. Now set $\mathscr{B} = \mathscr{C}(P,R)$ and for non-negative integers D define

$$H(\alpha;D) = \frac{1}{D+1}\left|\sum_{|x|\le D} e(\alpha x)\right|^2 = \sum_{|x|\le 2D}\left(\frac{2D+1-|x|}{D+1}\right)e(\alpha x).$$

Define $f(\boldsymbol{\alpha})$ as in section 5 and let

$$\widetilde{J}_{s,g,h}(\mathscr{B};\boldsymbol{D}) = \int_{\mathbb{U}^{g-h+1}} |f(\boldsymbol{\alpha})|^{2s} G(\boldsymbol{\alpha})\,d\boldsymbol{\alpha}, \qquad G(\boldsymbol{\alpha}) = H(\alpha_h;D_h)\cdots H(\alpha_g;D_g).$$

Then $J^*_{s,g,h}(\mathscr{B};\boldsymbol{D}) \le \widetilde{J}_{s,g,h}(\mathscr{B};\boldsymbol{D})$, because the latter quantity counts the solutions of (8.1) each with weight

$$w(\boldsymbol{d}) = \prod_{j=h}^{g} \max\left(0, \frac{2D_j+1-|d_j|}{D_j+1}\right). \tag{8.2}$$

Since $G(\boldsymbol{\alpha})$ is real and non-negative, we may follow the proof of Lemma 4.1 to bound $\widetilde{J}_{s,g,h}(\mathscr{B};\boldsymbol{D})$. We show the proof in some detail, as this method may have other applications.

Lemma 8.1. *Suppose $h, \tilde{g}, g, r, s$ are positive integers with*

$$g \ge \tilde{g} \ge h \ge 9, \quad t = \tilde{g}-h+1, \quad h \le r \le \tilde{g}, \quad s \ge 2t.$$

Further suppose that

$$0 \le D_j \le sP^j \quad (h \le j \le g), \qquad D_j = 0 \quad (h \le j \le \tilde{g})$$

and

$$R = P^\eta > g^2, \qquad |\mathscr{C}(P,R)| \ge P^{1/2}, \qquad P > (8s2^{g/s})^8.$$

Then

$$J^*_{s,g,h}(\mathscr{C}(P,R);\boldsymbol{D}) \le \max\Big[(8s)^{2s}(22t^2)^{2s/\eta}2^g P^{s(1+1/r)},$$
$$4g^{2t(1+1/(r\eta))}(P^{1/r}R)^{2s-2t+\frac12(r-h)(r-h+1)}2^g P^t J^*_{s-t,g,h}(\mathscr{C}(P^{1-1/r},R);\boldsymbol{E})\Big],$$

where $E_j = \lfloor \frac{2D_j}{P^{j/r}} \rfloor$ for $h \le j \le g$.

Sketch of proof. First, $J^*_{s,g,h}(\mathscr{B};\boldsymbol{D}) \le \widetilde{J}_{s,g,h}(\mathscr{B};\boldsymbol{D})$, and we follow the proof of Lemma 4.1 to bound $S_0 := \widetilde{J}_{s,g,h}(\mathscr{B};\boldsymbol{D})$. Define $S_1,\dots,S_4$ analogously, and consider the same four cases. When S_1 is the largest, we obtain

$$S_0 \le (8s)^{2s}\int_{\mathbb{U}^{g-h+1}} |f(\boldsymbol{\alpha};P^{1/r})|^{2s}G(\boldsymbol{\alpha})\,d\boldsymbol{\alpha} \le (8sP^{1/r})^{2s}\int_{\mathbb{U}^{g-h+1}} G(\boldsymbol{\alpha})\,d\boldsymbol{\alpha}.$$

By (8.2), the last integral is $\le 2^{g-h+1} \le 2^g$, so $S_0 \le 2^g(8sP^{1/r})^{2s}$. However, the hypotheses imply $S_0 \ge (P-1)^{s/2}$, giving a contradiction. When S_2 is the largest,

$$S_0 \le 4t^2 \int_{\mathbb{U}^{g-h+1}} |f(\boldsymbol{\alpha})^{2s-2}f(2\boldsymbol{\alpha})|G(\boldsymbol{\alpha})\,d\boldsymbol{\alpha}$$
$$\le 4t^2 S_0^{1-1/s}\left(\int_{\mathbb{U}^{g-h+1}} |f(2\boldsymbol{\alpha})|^{2s}G(\boldsymbol{\alpha})\,d\boldsymbol{\alpha}\right)^{\frac{1}{2s}}\left(\int_{\mathbb{U}^{g-h+1}} G(\boldsymbol{\alpha})\,d\boldsymbol{\alpha}\right)^{\frac{1}{2s}}.$$

By considering the underlying Diophantine equations, the first integral on the right is $\le S_0$, thus $S_0 \le 4t^2 S_0^{1-\frac{1}{2s}} 2^{g/(2s)}$, whence $S_0 \le (4t^2)^{2s} 2^g$. Again by the lower bound $S_0 \ge (P-1)^{s/2}$ and the assumed lower bound on P, this gives a contradiction. Therefore, $S_0 = 4\max(S_3, S_4)$.

When S_3 is largest, we obtain

$$\begin{aligned} S_0 &\le (8s)^2(8et^2)^{2/\eta} P^{1+1/r} \int_{\mathbb{U}^{g-h+1}} |f(\boldsymbol{\alpha})|^{2s-2} G(\boldsymbol{\alpha})\, d\boldsymbol{\alpha} \\ &\le (8s)^2(8et^2)^{2/\eta} P^{1+1/r} \left(\int_{\mathbb{U}^{g-h+1}} |f(\boldsymbol{\alpha})|^{2s} G(\boldsymbol{\alpha})\, d\boldsymbol{\alpha} \right)^{1-\frac{1}{s}} \left(\int_{\mathbb{U}^{g-h+1}} G(\boldsymbol{\alpha})\, d\boldsymbol{\alpha} \right)^{\frac{1}{s}} \\ &\le (8s)^2(8et^2)^{2/\eta} P^{1+1/r} S_0^{1-1/s} 2^{g/s}. \end{aligned}$$

Therefore $S_0 \le (8s)^{2s}(22t^2)^{2s/\eta} 2^g P^{s(1+1/r)}$.

If S_4 is the largest, we add a factor $G(\boldsymbol{\alpha})$ to each $X_i(\boldsymbol{\alpha})$ and $Y_i(\boldsymbol{\alpha})$ and obtain

$$S_0 \le 4(P^{1/r}R)^{2s-2t} \max_{P^{\frac{1}{r}} < q \le P^{\frac{1}{r}} R} W(q),$$

where $W(q)$ counts solutions of

$$\sum_{i=1}^{t} (x_i^j - y_i^j) + q^j \sum_{i=1}^{s-t} (u_i^j - v_i^j) = d_j \qquad (h \le j \le g)$$

each with weight $w(\boldsymbol{d})$. Since $d_j = 0$ for $h \le j \le \tilde{g}$, the argument in the proof of Lemma 4.1 implies that there are at most $g^{2t(1+1/(r\eta))} q^{(r-h)(r-h+1)/2} P^t$ possibilities for $\boldsymbol{x}, \boldsymbol{y}$ (note that here $t = \tilde{g} - h + 1$). Let $\mathscr{S}$ be the set of possible $\boldsymbol{x}, \boldsymbol{y}$ and put

$$F(\boldsymbol{\alpha}) = \sum_{(\boldsymbol{x},\boldsymbol{y})\in\mathscr{S}} e\left(\sum_{j=h}^{g} \alpha_j (x_1^j - y_1^j + \cdots + x_t^j - y_t^j) \right).$$

Putting $\widetilde{\boldsymbol{\alpha}} = (q^h \alpha_h, \dots, q^g \alpha_g)$, we obtain

$$\begin{aligned} W(q) &\le \int_{\mathbb{U}^{g-h+1}} |F(\boldsymbol{\alpha})| |f(\widetilde{\boldsymbol{\alpha}}; P/q)|^{2s-2t} G(\boldsymbol{\alpha})\, d\boldsymbol{\alpha} \\ &\le g^{2t(1+1/(r\eta))} q^{(r-h)(r-h+1)/2} P^t \int_{\mathbb{U}^{g-h+1}} |f(\widetilde{\boldsymbol{\alpha}}; P/q)|^{2s-2t} G(\boldsymbol{\alpha})\, d\boldsymbol{\alpha}. \end{aligned}$$

The integral on the right counts the solutions of

$$q^j \sum_{i=1}^{s-t} (u_i^j - v_i^j) = d_j \qquad (h \le j \le g),$$

each counted with weight $w(\boldsymbol{d})$. Since $q > P^{1/r}$, this is at most 2^g times the number of solutions of

$$\sum_{i=1}^{s-t}(u_i^j - v_i^j) = e_j \qquad (h \le j \le g),$$

with $u_i, v_i \in \mathscr{C}(P^{1-1/r}, R)$ and $|e_j| \le 2D_j/P^{j/r}$. This proves the lemma in the last case. □

In Lemma 8.1 it is common that there are more zeros among the numbers E_i than among the numbers D_i. Thus, as Lemma 8.1 is iterated, t steadily increases (if t reaches $g-h+1$, then one can apply the bounds from §4). This is the primary source of the improvement over Lemma 4.1, but the analysis of the exponents of P and the constants is much more complicated. The analysis becomes even more complex if repeat efficient differencing is used. By taking optimal parameters, using Lemma 8.1 in place of Lemma 4.1 has the potential to reduce B by about 0.09, or $\approx 2\%$.

Lastly, we indicate what is the limit of our method, i.e. the limit of what could be accomplished with Lemma 5.1. Assume now that the lower bound (1.5) for $J_{s,k}(P)$ is close to the truth, i.e. $J_{s,k}(P) \le C(k,s)P^s$ for $s \le \frac{1}{2}k(k+1)$. Assume also best possible upper bounds $J_{s,g,h}(\mathscr{B}) \le C(s,g,h)P^s$ for $s \le \frac{t}{2}(g+h)$, valid for any $\mathscr{B} \subset [1,P]$. Adopt the notations from section 5. With these assumptions, it turns out that the best choices for r, s, μ_1, μ_2 are given by

$$r = \frac{k(k+1)}{2}, \quad s = \frac{t(g+h)}{2}, \quad \mu_1 = \mu_2 = \mu = \frac{1}{6}.$$

Also, one takes ϕ very close to (and larger than) $\frac{1}{1-\mu}$ and γ very close to (and smaller than) $\frac{1}{1-\mu}$. Plugging these values into (5.22) yields

$$\lambda^2 E = \frac{2}{27} - \varepsilon,$$

where $\varepsilon \to 0^+$ as $\phi - \gamma \to 0$. An application of Lemma 7.3 (with $D = 27/2 + \varepsilon'$) gives Theorem 1 with a constant $B = \sqrt{2} + \varepsilon''$ (valid for $\sigma \ge \frac{15}{16}$), where $\varepsilon', \varepsilon''$ can be taken arbitrarily small.

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Appendix: Computer program listings

```
/* PROGRAM 1. exponents and constants in Vinogradov's integral for small k.
   Used to prove the second part of Theorem 3; written 12/12/2000 K. Ford  */
#include <stdio.h>
#define max(x,y) (((x)>(y))?(x):(y))
#define min(x,y) (((x)>(y))?(y):(x))
double newdel(k,r,del)       /* returns delta_0(k,r,del) */
     double k,r,del;
{
  double y, sqrt(),p,tkr;
  long j,jj;
  if ((r<4.0)  (r>k)) return(2.0*del);  /* invalid r */
  tkr = 2.0*k*r; y=2.0*del-(k-r)*(k-r+1.0);
  if ((y<0.0)(2.0*k/(tkr+y))<=1.0/(k+1.0)) return(del*2.0); /* invalid r */
  j = min((long) (0.5*(3.0+sqrt(4.0*y+1.0))), 9*r/10);
  p = 1.0/r;
  for (jj=j-1; jj>=1; jj--)
    p = 0.5/r+0.5*(1.0+(jj*jj-jj-y)/tkr)*p;
  return(del-k+0.5*p*(tkr-y));
}
main()
{
  long j,k,k0,k1,r,r0,r1,n,bestr,s;
  double kk,logk,del0,del1,sqrt(),log(),exp(),bestdel, goal, maxs, eta, om;
  double logH,logW,logC,k3,theta,thetamax;
  printf("enter k range : "); scanf("%ld %ld",&k0,&k1);
  maxs = 0.0;  thetamax=0.0;
  for (k=k0; k<= k1; k++) {
    kk=(double) k;
    logk=log(kk); k3 = kk*kk*kk*logk;
    om=0.5; for (j=1;j<=10;j++) om=1.5/(log(18.0*k3/om)-1.5);
    eta = 1.0+om;
    logW = (kk+1.0)*max(1.5+1.5/om, log(18.0/om*k3));
    del0 = 0.5*kk*kk*(1.0-1.0/kk);
    goal = 0.001*kk*kk;
    logH = 3.0*kk*logk+(kk*kk-4.0*kk)*log(eta); /* log(k^3k eta^(k^2-4k) */
    logC = kk*logk;                             /* upper bound for log(k!) */
    for (n=1;;n++) {
      r0 = (long) (sqrt(kk*kk+kk-2.0*del0)+0.5)-2; r1 = r0+4; /* r range */
      bestdel=kk*kk; bestr=-1;
      for (r=r0;r<=r1;r++) {
        del1=newdel(kk,(double) r,del0);
        if (del1<bestdel) { bestdel=del1; bestr=r; }
      }
      del1=bestdel; r=bestr;
      if ((del1 >= del0)  (r<r0)) exit(-1);
```

```
      logC += max(logH + 4.0*kk*n*log(eta),logW*(del0-del1));
      if (del1<=goal) {                                 /* reached goal */
        s=(long) ((n+(del0-goal)/(del0-del1))*kk+1);
        theta = logC/k3;
        printf("%4d:  s=%8.6f k^2   eta=%9.7f   theta=%10.8f\n",k,
            s/kk/kk,eta,theta);
        if ((s/kk/kk) > maxs) maxs=s/kk/kk;
        if (theta>thetamax) thetamax=theta;
        break;
      }
      del0=del1;
    }
  }
  printf("\n max s = %9.6fk^2  maxtheta=%10.8f\n",maxs,thetamax);
}
```

```
/* PROGRAM 2.  Find optimal parameters for use in bounding S(N,t) for the
   Riemann zeta function  : intermediate lambda.  For Lemma 5.3,
   lambda in [84,220].  By K. Ford 10/22/2001        */
#include <stdio.h>
#include <math.h>
long k,g,h,s,r,t, g0, h0,g1,h1,flag;
double mu1,mu2,xi,lam,lam1,lam2,D,sigma, Y, goal;
void calc(ex,c,pr)
     double *ex,*c; int pr;
{
  double kk,logk, k2, log(),exp(),pow(),floor(), ceil();
  double th,rr,ss,tt,gg,hh,rho,H,E1,E2,E3,m1,m2,Z0,Z1,reta,
     logC1,logC2,logC3,logC,dc;
  k=(long) (lam/(1.0-mu1-mu2)+0.000003);
  /* if (k<129) exit(-1); */
  kk=(double) k;
  logk=log(kk); k2=kk*kk;
  rho=3.21432; th=2.3291;
  if (k<=199)  { rho=3.21734; th=2.3849; }    /* 150 to 199 */
  if (k<=149)  { rho=3.22313; th=2.4183; }    /* 129 to 149 */
  r = (long) (rho*k2+1.0);
  rr=(double) r;  ss=(double) s;
  gg=(double) g;  hh=(double) h; tt=(double) t;
    /* calculate minimum H = Z1 + lam*Z2 */
  m1 = floor(lam/(1.0-mu1));
  m2 = floor(lam/(1.0-mu2));
  Z0 = 0.5*((m1*m1+m1)*(1.0-mu1)+(m2*m2+m2)*(1.0-mu2)-hh*hh+hh-(1.0-mu1-mu2)*
    (gg*gg+gg));
  Z1 = hh+gg-m1-m2-1.0;
  if (Z1<0.0) H = Z0 + lam2*Z1;
  else H=Z0 + lam1*Z1;   /* H is now the H' from Lemma 5.3 */
  reta = xi*pow(gg,1.5);   /* 1/eta */
  E1 = 0.001*k2;
  E2 = 0.5*tt*(tt-1.0)+hh*tt*exp(-ss/(hh*tt))+ss*ss/(2.0*tt*reta);
  E3 = log(Y*lam1*lam1)/(7.5*Y*lam1*lam1*lam1*lam1);
  *ex = (-E3 + (1.0/(2.0*rr*ss))*(H-mu1*E1-mu2*E2))*lam1*lam1;
```

```
  logC1=th*k2*kk*logk;
  logC2 = ss*ss/tt+10.5*xi*xi*tt*gg*gg*log(gg)*log(gg)/D;
  logC2 -= (ss*log(0.1*reta)*((reta+hh)*pow(1.0-1.0/hh,ss/tt)-h));
  logC3=1.04*reta*log(10.82*reta);
  logC = logC3/rr+(5.0*lam2*log(lam2)+logC1+logC2)/(2.0*rr*ss);
  *c = exp(logC)+1.0/kk;  /* constant for exponent ex */
  if (pr==1) {
    printf("%8.4f-%8.4f  %4d",lam1,lam2,k);
    if (g>0) printf(" %3d  %2d  %2d  %2d  %9.4f %7.4f\n",
    s,g-g0,h1-h,t,1.0/(*ex)+0.00005,*c+0.00005);
    else printf("\n");
  }
}
main()
{
  double E,lam8,lam9,r[9],tmp,maxex,con,maxcon,bestth,bestcon,bp[5000];
     /* bp[] are endpoints of intervals */
  long i,j,i0,w,n,m,maxm,bestg,besth, bests,s0,s1;
  mu1 = 0.1905; mu2 = 0.1603;
  goal=133.66;
  while (1) {
    printf("enter Y  : "); scanf("%lf",&Y);
    D = 0.1019*Y;
    printf("enter xi : "); scanf("%lf",&xi);
    printf("enter sigma : "); scanf("%lf",&sigma);
    if (sigma<0.0) flag=1; else flag=0;
    /* flag=1 means let the program find the best value of s */
    printf("enter lambda range: "); scanf("%lf %lf",&lam8, &lam9);
    if ((lam9<lam8)  (lam8<=80.0)  (lam9>=300.0)) continue;
    printf("    approx.\n");
    printf("  lambda range       k   s    a   b   t     exp      const\n");
    printf("----------------   ---- ---  --- --- ---  --------  --------\n");
    bp[1] = lam8; bp[2] = lam9; j=3;       /* make list of endpoints */
    i0 = (long) (lam9/(1.0-mu1-mu2))+10;
    for (i=1; i<=i0;i++) {
      w=(double) i;
      r[1]=w*(1.0-mu1);
      r[2]=w*(1.0-mu2);
      r[3]=(w-0.000003)*(1.0-mu1-mu2);
      for (m=1;m<=3;m++) if ((r[m]<lam9) && (r[m]>lam8)) bp[j++]=r[m];
    }
    n=j-1;      /* number of endpoints */
    for (i=1; i<=n-1; i++) for(j=i+1;j<=n;j++)   /* Bubble sort */
        if (bp[j]<bp[i]) { tmp=bp[i]; bp[i]=bp[j]; bp[j]=tmp; }
    maxex=0.0;  /* maximum exponent of N */
    maxcon = 0.0;  /* maximum constant */
    for (j=1; j<=n-1; j++) {
      lam = 0.5*(bp[j]+bp[j+1]);   /* midpoint of interval */
      lam1=bp[j]; lam2=bp[j+1];    /* endpoints */
      g0 = (long) (lam/(1.0-mu1)+1.0); g1=g0+1; /* g range */
      h1 = (long) (lam/(1.0-mu2)); h0=h1-1;     /* h range */
      bestg=-1; besth=-1; bestth=1.0e20; bestcon=1.0e40;
```

```
      for (g=g0;g<=g1;g++) for (h=h0;h<=h1;h++) {
       t=g-h+1;
       if ((g>=100) && ((double) g <= 1.254*lam1)) {  /* condition (5.16) */
         if (flag==0) {
           s0=(long) (sigma*h*t+1.0); s1=s0;
         }
         else {
           s0=h*(t-1)/4;
           s1=h*t/2;
         }
         for (s=s0; s<=s1; s++) {
           calc(&E,&con,0);
           if ((E>0.0) && (1.0/E < goal) && (con<bestcon)) {
             /* look for best constant such that 1/exponent < goal */
             bestth=1.0/E; bestg=g; besth=h; bests=s;
           }
         }
       }
      }
      g=bestg; h=besth; t=g-h+1;
      s=bests;
      calc(&E,&con,1);
      if (1.0/E>maxex) maxex=1.0/E;
      if (con>maxcon) maxcon=con;
    }
    printf("  max. ex: %10.6f     max. const.: %10.6f\n",maxex,maxcon);
  }
}
```

```
/* PROGRAM 3.  find optimal parameters for use in bounding S(N,t)
   for small lambda; Section 6.  Written by K. Ford 10/20/2001 */
#include <stdio.h>
#include <math.h>
#define max(x,y) (((x)>(y))?(x):(y))
#define min(x,y) (((x)<(y))?(x):(y))
long k,n0;
double kk, logk, logk1, pi, eta, logeta, L32, lam, lam4, lkf,k3,logA,B,C;
double Delta[10000], logC[10000]; /* Delta and log of constants */
double log(), exp(), pow(), sqrt();
/* #define DEBUG  */
double logV(double w)   /* log(V(w))  */
{
 if (w==1.0) return(k3);
  if ((w<=0.5)&&(w>0.0)) return(max(1.5+1.5/w,k3+log(3.0/w)));
  exit(-1);
}
double F(double w)
{
  return((1.0+w)*exp(logA/B)-exp(logV(w)*C/B));
}
double bestomega(int n)  /* best omega value for Lemma 6.5 */
{
```

```
  double w0,w1,w2;
  B = kk*kk-Delta[n];            /* exponent of (1+w) */
  C = Delta[n];                  /* exponent of V     */
  if (F(1.0)<=0.0) return(1.0);  /* take w=1          */
  if (F(0.5)<=0.0) {             /* take w=1 or 1/2   */
    if (exp(logV(0.5)*C/B)<2.0*exp(logA/B)) return(0.5);
    else return(1.0);
  }                              /* solve F(w)=0 */
  w0=0.5; w1=0.2; while (F(w1)>=0.0) w1*=0.5;
  while (((w0-w1)/w1)>=0.0000001) {
    w2=0.5*(w0+w1);
    if (F(w2)>0.0) w0=w2; else w1=w2;
  }
  return(w1);
}
void calcparm()   /* calculate Delta_n and C_n */
{
  long n1,n,i;
  double f, s, logU, omega, logM1, logM2, AA, BB;
  kk=(double) k;
  logk=log(kk);
  logk1=log(kk-1.0);
  k3=3.0*logk+log(6.0*logk);  /* log(6k^3 log k) */
  lkf=0.0; for (i=2;i<=k;i++) lkf += log(((double) i));  /* log(k!) */
  logA = 3.0*logk+lkf+log(4.0);  /* log(4k^3 k!)   */
  logeta=log(eta);
  L32=log(32.0)-lkf;
  n1 = (long) (2.6*kk*logk+50);
  if (n1>=9999) n1=9998;  /* calculate constants up to n=n1    */
  for(i=1;i<=n0;i++) {    /* use trivial bound for 1<= n<= n0  */
    Delta[i] = 0.5*kk*(kk-1.0);
    logC[i] = lkf;
  }
  f = 1.0-1.0/kk;
  for (n=n0+1; n<=n1+1; n++) Delta[n]=f*Delta[n-1];
  for (n=n0; n<=n1; n++) {
    s=kk*n;
    omega=bestomega(n);
    logM1 = max(logV(omega)*C,logA+B*log(1.0+omega));
                     /* M1=multiplier for constant in Lem. 6.5 */
    if (k>= 9) {     /* Lemma 6.7 only for k>=9  */
      AA =(kk*kk-Delta[n])*logeta+2.0*kk*log(s+kk)+L32;
      logU = (2.0*kk-2.0+(2.0*s+2.0)*logk1)/
        (2.0*s+2.0-0.5*kk*(kk+1.0)+Delta[n+1]);
      if (logU<logk) logU=logk;
      BB=Delta[n]*logU;
      logM2 = max(AA,BB);   /* M2=multiplier for constant in Lemma 6.7 */
    }
    else logM2=1.0e40;
    logC[n+1] = logC[n] + min(logM1,logM2);
#ifdef DEBUG
    printf("  logM1=%f  logM2=%f  logC[%d]=%f\n",logM1,logM2,n+1,logC[n+1]);
```

```
#endif
  }
}
int exponent(n,c,pr)      /* from Lem. 6.3, 6.4 */
  int n,pr; double *c;    /* return constant in 'c' */
{
  double s,goal,logd,c1,e,mu,log(),pow(),exp();
  lam=kk-1.0; if (k==4) lam=lam4;  /* lower limit of lambda */
  mu=1.0-lam/(kk+1.0);             /* largest mu */
  s=kk*n;
  logd = log(4.0) + 0.5/s*(logC[n]+lkf+kk*log(2.0*kk*pi));
  logd = log(exp(logd)+2.0);  /* add 2 */
  goal = 133.66*lam*lam;      /* goal for denominator */
  e = (1.0-(1.0+Delta[n])*mu)/(2.0*s);
  if (e<1.0/goal) return(-1);   /* exponent not good enough */
  *c = exp(logd/e/goal);
  if (((*c) <= 10000.0) && (pr==1))
    printf("n=%6d   1/(e lam^2)=%8.2f  c=%e\n",n, 1.0/e/(kk-1.0)/(kk-1.0),*c);
  return(0);
}
main()
{
  double log(),bestc,c,e,mu, CC[200];
  long bestn,bestn0,n, n2, i,k1,k2,j,nn[200], n00[200], n01, n02;
  pi=3.1416;  /* good enough upper bound */
  printf(" k range : "); scanf("%ld %ld",&k1,&k2);
  if (k1<4) exit(0);
  if (k1==4) {
    printf("enter lower bound on lam for k=4 : ");
    scanf("%lf",&lam4);
  }
  /* printf(" n0 range : "); scanf("%ld %ld",&n01, &n02); */
  for (k=k1; k<=k2; k++) {
    if (k<=13) eta=1.308;
    else if (k<=32) eta=1.2609;
    else eta=1.12766;
    bestn0=0; bestn=0; bestc=1.0e40;
    for (n0=1; n0<=2*k; n0++) {
      calcparm();
      n2 = (long)  (kk*2.5*logk) + 50;
      for (n=k+1;n<=n2;n++) {
       if (exponent(n,&c,0)==0) {
         if (c<bestc) { bestc=c; bestn=n; bestn0=n0; }
       }
      }
      if (bestn<1) bestc=-99.99;
    }  /* for n0 */
    if (bestn0<1) CC[k]=-99.99;
    else {
#ifdef DEBUG
      for (n=bestn-25; n<=bestn+5 ; n++) exponent(n,&c,1);
#endif
```

```
      nn[k]=bestn; CC[k]=bestc+0.00005; n00[k]=bestn0;
      printf("k=%d   lambda: %d - %d    n0=%d  n=%d  c=%8.5f\n",
          k,k-1,k,bestn0,bestn,bestc+0.000005);
    }
  }    /* for k */
  nn[k2+1]=999; CC[k2+1]=99.999;  /* print in TeX tabular format */
  i = (k1+k2)/2-k1+1;
  for (j=k1; j<=(k1+k2)/2; j++) {
    if (j==4) printf("&& %3.1f",lam4);
    else printf("&& %3d",j-1);
    printf("--%-2d & %2d & %3d & %3d & %7.4f &&",j,j,n00[j],nn[j],CC[j]);
    printf(" %2d--%-2d & %2d & %3d & %3d & %7.4f &\\cr\n",j+i-1,j+i,j+i,
      n00[j+i],nn[j+i],CC[j+i]);
  }
}
```

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